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Matrix factorisations and D-branes on K3

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Abstract: D-branes on K3 are analysed from three different points of view. For deformations of hypersurfaces in weighted projective space we use geometrical methods as well as matrix factorisation techniques. Furthermore, we study the D-branes on the $T^4/\mathbb{Z}_4$ orbifold line in conformal field theory. The behaviour of the D-branes under deformations of the bulk theory is studied in detail, and good agreement between the different descriptions is found.

Keywords: D-branes, Conformal Field Models in String Theory, Topological Strings.

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1. Introduction

D-branes in models with $N = (2, 2)$ worldsheet supersymmetry have been discussed in recent years from various points of view. Prominent examples for such theories include non-linear sigma-models whose targets are Kähler manifolds, Gepner models and Landau-Ginzburg models. Boundary conditions can be formulated in terms of the $N = (2, 2)$ supersymmetry algebra, and one is usually interested in boundary conditions that preserve half of the supersymmetry. As is well known, there are two different classes of such supersymmetry preserving boundary conditions, which are related by mirror symmetry and are called A-type and B-type. In the non-linear sigma-model, A-type boundary conditions correspond to D-branes wrapping (special) Lagrangian cycles, whereas B-type boundary conditions describe holomorphic branes [1]. In the Gepner model, which provides a rational conformal field theory description in the small volume regime of certain Calabi-Yau compactifications, A-type and B-type D-branes can be constructed as explicit boundary states with appropriate gluing conditions for the generators of the symmetry algebra. Finally, in the Landau-Ginzburg model, A-type D-branes correspond to Lagrangian submanifolds that are mapped by the superpotential to straight lines [2, 3], whereas B-type D-branes can be described in terms of matrix factorisations of the superpotential. The study of matrix factorisations was initiated in this context by Kontsevich, who proposed that there is a B-type D-brane for any factorisation of the superpotential $W(x_i) = E(x_i)J(x_i)$ in terms of matrices E and J with polynomial entries. One then associates a BRST operator of the form

$$Q = \begin{pmatrix} 0 & J \\ E & 0 \end{pmatrix} \tag{1.1}$$

to this factorisation and determines the (topological) open string spectrum as the BRST cohomology of this operator [4 – 9].

In this paper, we shall analyse supersymmetric D-branes on K3 surfaces using matrix factorisation and conformal field theory techniques. K3 surfaces are special since they actually preserve $N = (4, 4)$ supersymmetry, not just $N = (2, 2)$. The relation between the geometrical and conformal field theory description of closed strings on K3 has been analysed in some detail in [10]; here we shall concentrate mainly on open strings. A-type and B-type boundary conditions can be formulated as usual once a particular $N = 2$ subalgebra is chosen. Geometrically, all supersymmetric D-branes are holomorphic with respect to some complex structure. For a given complex structure, on the other hand, not all of these D-branes are holomorphic and the spectrum of holomorphic D-branes thus depends on the actual point in moduli space. One expects to find D0 and D4 branes at any point in moduli space, but the rank of the Picard lattice, that determines the number of holomorphic curves and hence D2 branes, varies.

An explicit conformal field theory description of string theory on K3 is only available at rather specific points in the moduli space, but the matrix factorisation description is, in principle, available for a much larger subspace of the moduli space. Thus the latter approach is very well suited to study the spectrum of B-type D-branes for generic points in the moduli space. In this paper we shall study some aspects of matrix factorisations

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for K3s that can be described as hypersurfaces in weighted projective space, where we restrict to the case of Fermat type polynomials and their perturbations. At the maximally symmetric point, string theory on this surface has a conformal field theory description in terms of a Gepner model.

At this Gepner point we can easily construct factorisations by tensoring the usual single factor and the permutation factorisations together.[1] We can then analyse how these B-type D-branes can be deformed as one perturbs the superpotential. For certain factorisations (in particular, the tensor product factorisation and single transposition factorisations) we will be able to show that they can be extended over the whole moduli space of deformations. This can be done very explicitly by writing down the corresponding factorisations for arbitrarily deformed superpotentials (see section 3). This result has a very nice geometrical interpretation: these factorisations account precisely for those holomorphic D2-branes that come from the embedding space or from resolving singular points of the embedded manifold, and are generically present for K3s that arise as hypersurfaces in weighted projective space.

On the other hand, we can also prove that certain factorisations *cannot* be deformed. This can be shown by studying the infinitesimal deformations following [12, 53]. In particular, we shall show that a necessary condition for a deformation not to be obstructed is that the brane is uncharged under the RR ground state corresponding to the deforming polynomial. The fact that factorisations are generically obstructed is also in good agreement with the geometrical results.

In order to compare these results with what can be analysed in conformal field theory we can make use of the fact that there is an interesting subspace of the moduli space of such K3 surfaces for which we have an explicit conformal field theory description. Indeed, the deformations of the quartic surface in $\mathbb{P}_3$ by two particular bulk fields

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 + a x_1^2 x_2^2 + b x_3^2 x_4^2 = 0 \tag{1.2}$$

describes [13] the 2-parameter space of toroidal $\mathbb{Z}_4$ orbifold K3s. For this subspace of the moduli space one should thus be able to relate the different matrix factorisations with explicit conformal field theory constructions of D-branes; this will be done in section 4.

In the orbifold description it is straightforward to see that the space of B-type RR charges is 22-dimensional for any choice of the orbifold parameters a and b,[2] and it is in principle not difficult to construct the relevant D-branes in the orbifold conformal field theory. On the matrix factorisation side it is likewise not difficult to construct the relevant 22 factorisations that account for all of these charges at the Gepner point (where $a = b = 0$). However, we can show that not all of them can be extended to arbitrary a, b. In fact, given our general results about obstructions, it is clear that for the factorisations that are charged under the RR ground states corresponding to $x_1^2 x_2^2$ or $x_3^2 x_4^2$ this will not be possible.

[1] Unlike the situation for $3d$ Calabi-Yau's [11] these factorisations do not in general account for all B-type RR charges. This is a consequence of the fact that for K3 the middle dimensional cycles are 2-dimensional. Given the relation between matrix factorisations and geometry, one should thus not expect to obtain the charges of all holomorphic 2-cycles by these constructions.

[2] The relation between the two theories involves mirror symmetry. These D-branes are therefore A-type from the point of view of the orbifold theory.

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Furthermore, we can show that this is the only real obstruction: we have identified a set of factorisations that account for 20 RR charges and that can be extended for arbitrary a and b.

This apparent obstruction has in fact a very nice interpretation in terms of the orbifold conformal field theory. At least some of the relevant D-branes that carry these charges stretch diagonally across the two T^2s at 45 degrees. Varying the parameters a and b corresponds then to changing the radii of the two tori (as well as switching on a B-field). The structure of the 45 degree D-brane[3] then depends crucially on the relative radii: if their ratio is rational, the brane will have finite length, but it will wind infinitely many times around the torus if the ratio is irrational [14]. Thus the boundary state depends in a very discontinuous manner on the parameters of the closed string theory. (This is also familiar from the analysis of the $N = 0$ and $N = 1$ D-branes on a single circle [15 – 17].) This explains why the corresponding matrix factorisation description cannot depend on the deformation parameters in a simple analytic way.

The paper is organised as follows. In section 2 we review some background material about D-branes on K3 surfaces from a geometric point of view. In particular, we describe how the rank of the Picard lattice is always at least 1 for K3 surfaces that are embedded in weighted projective space. In section 3 we briefly review the matrix factorisation approach to D-branes in Landau Ginzburg models. We then discuss the behaviour of D-branes under bulk perturbations in this language, and explain, in particular, how the generic rank of the Picard lattice can be understood from this perspective. We also study the quartic in $\mathbb{P}_3$ and the above deformations in detail. Finally, in section 4 we discuss the orbifold theory $T^4/\mathbb{Z}_4$ and its D-branes. We explain some parts of the correspondence between the boundary states of the orbifold theory and the matrix factorisations of the Landau-Ginzburg description. Finally we study their deformations in the orbifold theory and explain why certain deformations are obstructed. We have included an appendix in which some of the more technical aspects of the orbifold description and its dictionary to the Gepner model are explained in detail.

2. BPS D-branes on K3 surfaces

Let us begin by explaining some generalities about D-branes on K3 surfaces [1, 18, 19]. On K3 we are in the special situation that there is extended $N = (4, 4)$ supersymmetry. The $N = 4$ algebra is an extension of the usual $N = 2$ superconformal algebra, where the $u(1)$ current of the $N = 2$ theory is enhanced to an $\widehat{su}(2)_1$ algebra; the additional generators are the spectral flow operators (by one unit), which have conformal weight 1 for $c = 6$.

From the point of view of the extended $N = (4, 4)$ symmetry there is therefore some freedom in how to choose the $u(1)$ generator of the $N = 2$ algebra inside the $\widehat{su}(2)_1$ algebra of the $N = 4$ algebra. This is precisely the freedom of choosing a Cartan torus for the $SU(2)$ group. Each $N = 2$ subalgebra determines uniquely an $u(1)$ subalgebra of the $\widehat{su}(2)_1$, but

[3] One may also consider modifying the angle with which the D-brane stretches across the tori. However, the resulting D-brane will then typically not satisfy the correct $N = 2$ gluing conditions any more.

the converse is not true [13, 21]. Once we have identified in addition a particular $N = (2, 2)$ subalgebra, we can formulate A and B type boundary conditions as usual. However, it is clear that the distinction between A-type and B-type branes depends on the choice of the particular $N = (2, 2)$.

Mirror symmetry corresponds algebraically to flipping the sign of the $u(1)$ current of the left moving supersymmetry algebra. Obviously, this operation requires that a particular $N = (2, 2)$ structure has been picked. The mirror operation can then be viewed as a rotation of the Cartan torus (for the left movers).

Geometrically, a $K3$ surface S is a hyperkaehler manifold with $H^2(S, \mathbb{Z}) = 22$. With respect to the usual intersection product, the resulting lattice is even and self-dual, and has signature $(+)^3, (-)^{19}$. A hyperkaehler structure is determined by the positive 3-plane spanned by the periods of the three hyperkaehler forms in that lattice. Once a compatible complex structure is chosen, this three-plane has an orthogonal decomposition into the line generated by ω (the Kaehler form), and the plane spanned by the real and imaginary components of the holomorphic 2-form $\Omega = x + iy$. A change of complex structure amounts to rotating the 2-plane spanned by the vectors x, y. In the context of string theory, the moduli space contains in addition the B-field, and the full moduli space takes the form of a Grassmannian parametrising 4-planes in $\mathbb{R}^{4,20}$. A decomposition of the positive 4-plane into two orthogonal 2-planes then amounts to fixing the complex structure, a Kaehler class and a B-field. In [20], the four 2-forms (three hyperkaehler forms and the B-field) have been combined into a single quaternionic 2-form. Mirror symmetry, which interchanges the complex structure with the complexified Kaehler structure, acts in this language as a quaternionic rotation of the positive 4-plane.

Comparing with the conformal field theory description, the choice of a decomposition of the positive 4-plane into two perpendicular 2-planes amounts to the choice of an $N = (2, 2)$ subalgebra inside the $N = (4, 4)$. The two $SU(2)$ enhancing the $N = (2, 2)$ to $N = (4, 4)$ can be understood as the freedom to rotate the two 2-planes.

Geometrically, B-type D-branes correspond to holomorphic branes, whereas A-type branes wrap (special) Lagrangian submanifolds. In the case of $K3$, B-type branes can have dimension $0, 2, 4$, whereas A-type branes are always 2-dimensional. Some of the 22 2-cycles will thus be wrapped by A-type branes, and some by B-type branes, but the decomposition into A-type and B-type branes depends, of course, on the chosen complex structure. For example, the quaternionic rotation that induces mirror symmetry exchanges holomorphic and Lagrangian cycles. The action of mirror symmetry on the D-branes can also be understood from the point of view of [24], where mirror symmetry was formulated for elliptic fibrations with a section as T-duality on the fiber. In the K3 context, this point of view has been used to extend mirror symmetry to the open string sector in [25]. Homological mirror symmetry has been proven for the quartic surface in [26].

2.1 B-type branes and the Picard lattice

As we have explained above, supersymmetric 2-cycles on K3 are holomorphic curves with respect to *some* complex structure. If a 2-cycle is holomorphic with respect to a given

complex structure, it can be wrapped by a D-brane that is B-type with respect to the corresponding $N = (2, 2)$ subalgebra. In the following we review some background material from [27].

The 2-cycles are naturally elements of $H_2(S, \mathbb{Z})$, or, using duality, of $H^2(S, \mathbb{Z})$. Holomorphicity imposes that the dual 2-form is in fact in $H^{1,1}(S)$, and the Picard lattice is thus

$$\mathrm{Pic}(S) = H^2(S, \mathbb{Z}) \cap H^{1,1}(S) \,. \tag{2.1}$$

The rank of the Picard lattice is usually denoted by ρ. Generically, K3 surfaces will have $\rho = 0$, meaning that no B-type 2-branes are compatible with the given holomorphic structure. However, in this paper we will always consider special geometric points at which the rank of the Picard lattice is enhanced or even maximal.

We are particularly interested in the case where S is a hypersurface described by a Fermat polynomial in a weighted projective space. In such a case, there is a correspondence [28] between the non-linear sigma model on the hypersurface and the Landau Ginzburg model with a superpotential that formally equals the polynomial appearing in the hypersurface equation. To be more precise, the hypersurface in $\mathbb{P}_{(w_1,w_2,w_3,w_4)}[H]$, with $H = \sum w_i$

$$x_1^{k_1+2} + x_2^{k_2+2} + x_3^{k_3+2} + x_4^{k_4+2} = 0 \,, \tag{2.2}$$

where $k_i + 2 = H/w_i$, $H = \mathrm{lcm}\{k_i + 2\}$, corresponds to the Landau Ginzburg orbifold model with superpotential

$$W = x_1^{k_1+2} + x_2^{k_2+2} + x_3^{k_3+2} + x_4^{k_4+2} \,. \tag{2.3}$$

In this equation the x_i denote chiral superfields of charge $q_L = q_R = 1/(k_i + 2)$; our notation will not distinguish between the chiral fields of the Landau-Ginzburg model and the coordinates of the projective space in the geometric description. The orbifold $\mathbb{Z}_H$ acts by phase multiplication on the chiral superfields $x_i \mapsto e^{\frac{2\pi i}{k_i+2}} x_i$; this orbifold projects onto integer $U(1)$ charges of the theory. Altogether, there are 14 different examples corresponding to Fermat polynomials in weighted projective space, which we list in table 1. These models also have a description in terms of rational conformal field theory, namely as the tensor product of 4 $N = 2$ minimal models at levels k_i, modulo an integer charge projection $\mathbb{Z}_H$. In terms of conformal field theory, it is straightforward to see that the space of B-type RR charges is 22-dimensional for each of these models. By standard conformal field theory arguments one should therefore expect that the corresponding B-type D-branes exist, and thus that the rank of the Picard lattice is maximal for all of these points.

The condition that S can be written as a hypersurface in weighted projective space imposes constraints on the allowed complex structure deformations, and thus increases the generic rank of the Picard lattice [27]. For example, in the case of the quartic in $\mathbb{P}_3$

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 + \cdots = 0 \tag{2.4}$$

there is at least one holomorphic curve at any point in the complex structure moduli space, namely the intersection of the quartic polynomial with any hyperplane. This phenomenon

Projective space	$W(x_1, x_2, x_3, x_4)$	Minimal model	RS	RS,ST
$\mathbb{P}_{(1,1,1,1)}[4]$	$x_1^4 + x_2^4 + x_3^4 + x_4^4$	$(2, 2, 2, 2)$	3	3
$\mathbb{P}_{(1,1,1,3)}[6]$	$x_1^6 + x_2^6 + x_3^6 + x_4^2$	$(4, 4, 4, 0)$	3	3
$\mathbb{P}_{(1,1,2,2)}[6]$	$x_1^6 + x_2^6 + x_3^3 + x_4^3$	$(4, 4, 1, 1)$	4	6
$\mathbb{P}_{(1,1,2,4)}[8]$	$x_1^8 + x_2^8 + x_3^4 + x_4^2$	$(6, 6, 2, 0)$	4	5
$\mathbb{P}_{(1,2,2,5)}[10]$	$x_1^{10} + x_2^5 + x_3^5 + x_4^2$	$(8, 3, 3, 0)$	4	8
$\mathbb{P}_{(1,1,4,6)}[12]$	$x_1^{12} + x_2^{12} + x_3^3 + x_4^2$	$(10, 10, 1, 0)$	4	4
$\mathbb{P}_{(1,2,3,6)}[12]$	$x_1^{12} + x_2^6 + x_3^4 + x_4^2$	$(10, 4, 2, 0)$	6	9
$\mathbb{P}_{(1,3,4,4)}[12]$	$x_1^{12} + x_2^4 + x_3^3 + x_4^3$	$(10, 2, 1, 1)$	6	12
$\mathbb{P}_{(2,3,3,4)}[12]$	$x_1^6 + x_2^4 + x_3^4 + x_4^3$	$(4, 2, 2, 1)$	6	14
$\mathbb{P}_{(1,2,6,9)}[18]$	$x_1^{18} + x_2^9 + x_3^3 + x_4^2$	$(16, 7, 1, 0)$	6	8
$\mathbb{P}_{(1,4,5,10)}[20]$	$x_1^{20} + x_2^5 + x_3^4 + x_4^2$	$(18, 3, 2, 0)$	8	12
$\mathbb{P}_{(1,3,8,12)}[24]$	$x_1^{24} + x_2^8 + x_3^3 + x_4^2$	$(22, 6, 1, 0)$	8	10
$\mathbb{P}_{(2,3,10,15)}[30]$	$x_1^{15} + x_2^{10} + x_3^3 + x_4^2$	$(13, 8, 1, 0)$	10	14
$\mathbb{P}_{(1,6,14,21)}[42]$	$x_1^{42} + x_2^7 + x_3^3 + x_4^2$	$(40, 5, 1, 0)$	12	12

Table 1: The 14 different K3 that correspond to Fermat polynomials in weighted projective space. The last two entries denote the rank of the charge lattice spanned by RS-branes and by RS and single transposition branes. As argued below in section 3.3.2, the last entry minus two (two charges correspond to D0- and D4-branes) should equal the rank of the Picard lattice at a generic point in the complex structure moduli space of the corresponding surface.

generalises immediately to all hypersurface equations in weighted projective space, where one can always consider the intersection with a hyperplane. In some examples the generic rank of the Picard lattice may be enhanced even further. Consider for example the model $\mathbb{P}_{(1,1,2,2)}[6]$. The embedding weighted projective space has a $\mathbb{Z}_2$ orbifold singularity with fixed point $(0, 0, x_3, x_4)$. The $\mathbb{Z}_2$ singularity is resolved by an exceptional $\mathbb{P}_1$. It intersects with the hypersurface in the 3 points that are defined by the equations $x_1 = x_2 = 0$ and $x_3^3 + x_4^3 = 0$. This enhances the rank of the generic Picard lattice by 3. Altogether, the rank of the Picard lattice is therefore 4 at generic points in the complex structure moduli space for this example. Note that there is one 2-cycle that is inherited from the embedding space: it corresponds to the combination of the three spheres, which is invariant under $x_3 \mapsto \exp(2\pi i/3)x_3$, which permutes the 3 singular points on the hypersurface. There are therefore 4 different brane charges, two D2, the D0 and D4 that the hypersurface inherits directly from the embedding space.

More generally, whenever two weights have a greatest common divisor m by which the other two weights are not divisible, the embedding projective space acquires an orbifold singularity which locally has the form $\mathbb{C}^2/\mathbb{Z}_m$. Its resolution requires $m - 1$ $\mathbb{P}_1$s whose intersection pattern is given by the A_{m-1} Dynkin diagram. This means that any such singularity contributes $m - 1$ 2-brane charges to the charge lattice that can be obtained by pull back from the embedding projective space. To determine the contribution to the Picard lattice of the hypersurface, one has to take into account that, as in the example above, the hypersurface might intersect the exceptional set in several points.

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Each of them gives a contribution to the Picard lattice. We will interpret these general features of the Picard lattice from the matrix factorisation point of view in section 3.

2.2 The orbifold line

Generically, the points in moduli space where a exact conformal field theory description is known are isolated. For example, for the above theories we only have a conformal field theory description (namely a Gepner model) for the unperturbed superpotential. There is, however, one interesting exception to this: since the $(2)^4$ model is in fact equivalent to the $\mathbb{Z}_4$ toroidal orbifold [10], there is a two-parameter family of orbifold theories all of which describe K3. The corresponding subspace of the moduli space has recently been identified to be [13]

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 + a x_1^2 x_2^2 + b x_3^3 x_4^2 = 0\,. \tag{2.5}$$

The orbifold theory will be described in more detail in section 4; the detailed mapping between a and b and the relevant parameters of the orbifold theory was given in [29, 13].

For this subspace of the moduli space we therefore have a good understanding of both the conformal field theory and the matrix factorisation approach. We should thus be able to compare the results from both points of view. The matrix factorisation description will be given in the following section, where we will in particular show that certain D-branes are obstructed against modifying the bulk parameters a and b. In section 4 we will identify the corresponding boundary states in the orbifold conformal field theory and reproduce these obstructions also from that point of view.

3. The matrix factorisation point of view

In this section we shall analyse the above theories from the matrix factorisation perspective. This approach was proposed in unpublished form by Kontsevich, and the physical interpretation of it was given in [4–9], for a review see also [30]. We shall first collect very briefly some basic facts about matrix factorisations that we shall need later on.

3.1 Fundamentals

Kontsevich has proposed that D-branes in a Landau-Ginzburg models are given by matrix factorisations of the superpotential,

$$Q^2 = W \cdot \mathbf{1}\,, \tag{3.1}$$

where Q is a square matrix with polynomial entries that satisfies

$$\sigma Q + Q\sigma = 0\,. \tag{3.2}$$

If we choose the grading operator σ to be diagonal, Q is of the form

$$Q = \begin{pmatrix} 0 & J \\ E & 0 \end{pmatrix} \quad \text{with} \quad EJ = JE = W \cdot \mathbf{1}\,. \tag{3.3}$$

Two factorisations (E, J) and (E', J') are considered equivalent if they are related by a similarity transformation with invertible matrices with polynomial entries,

$$E' = U_1 E U_2^{-1}\,, \qquad J' = U_2 J U_1^{-1}\,. \tag{3.4}$$

The spectrum of open strings between D-branes determined by factorisations (E, J) and (E', J') is then given by the cohomology of the boundary BRST operator Q. From a physics point of view, the factorisation condition can be derived by varying the Landau-Ginzburg action and cancelling the boundary terms [4, 5]. This analysis also confirms that Q is the correct boundary BRST operator. The proposal got further support by relating the results from the matrix factorisation perspective with those obtained in conformal field theory; in particular, this was done for the $N = 2$ minimal models in [5, 7, 31] and for tensor products of minimal models in [32, 33]. Finally, the matrix factorisation results for toroidal theories were shown to be in agreement with geometrical expectations [34 – 36].

We are particularly interested in Landau-Ginzburg orbifolds of the form (2.3). In this situation, the orbifold group $\mathbb{Z}_H$ gives an additional finer grading. This grading correspond to the choice of a representation γ_M such that Q satisfies

$$\gamma_M\, Q(\omega^{w_i} x_i)\, \gamma_M^{-1} = Q(x_i)\,, \tag{3.5}$$

where $\omega = e^{\frac{2\pi i}{H}}$. There are H different choices for γ_M that are labelled by M.

Given a matrix factorisation Q, the charge of the corresponding D-brane under the RR ground states can be calculated using the formulas derived in [6, 37, 38]. RR ground states in Landau-Ginzburg orbifolds arise both from the twisted and untwisted sector and can be counted using the techniques of [39]. The RR ground states from the untwisted sector correspond to polynomials in the Landau-Ginzburg fields and have the property that the $U(1)$ charges of the left and right moving part are equal, $q_L = q_R$. In the simplest case, where the weights do not have divisors (such as for the quartic in $\mathbb{P}_3$), there is exactly one RR ground states from each twisted sector.

For the general case, let $n = 0, \ldots, H-1$ label the twisted sectors. In each sector, consider only the untwisted fields x_i such that $nw_i/H \in \mathbb{Z}$, and set all twisted fields to zero. Let $\phi_n^\alpha = \prod_i (x_i)^{\alpha_i}$ be a basis of the untwisted chiral ring $\mathcal{J}_n = \mathbb{C}[x_i]/\partial W_n$ such that $\sum_i \alpha_i w_i/H = \sum_i (\frac{1}{2} - w_i/H)$. The RR ground states $|n;\alpha\rangle$ are obtained by acting with ϕ_n^α on the unique state $|n;0\rangle$. (For $n = 0$ this representation corresponds to (4.7) — see below.) Note that not all of the RR ground states obtained in this way survive the orbifold projection, which has to be imposed for all twists. The RR charge of Q with respect to a surviving RR ground state $|n;\alpha\rangle$ is given by [6, 37]

$$\mathrm{ch}(Q)(|n;\alpha\rangle) = \frac{1}{(2\pi i)^{r_n}} \oint dx_1 \ldots dx_{r_n} \frac{\phi_n^\alpha \mathrm{Str}[\gamma_M^n \partial_1 Q_n \ldots \partial_{r_n} Q_n]}{\partial_1 W_n \ldots \partial_{r_n} W_n}\,. \tag{3.6}$$

Here r_n is the number of untwisted fields, and W_n and Q_n are the superpotential and the factorisation with all twisted fields set to zero. The supertrace is the trace taken with the grading operator σ included, i.e. $\mathrm{Str}[\cdot] = \mathrm{tr}[\sigma\cdot]$.

In the context of the correspondence between Landau-Ginzburg models and Calabi-Yau manifolds, one would expect that one can associate to any matrix factorisation an element of the derived category of coherent sheaves of the Calabi-Yau manifold. The derived category of coherent sheaves and the category of graded matrix factorisations have to be equivalent since both are believed to describe the topological category of B-type branes [40, 37], which is supposed to decouple from the Kaehler moduli. One way to investigate this correspondence would be to analyse matrix factorisations in the context of the linear sigma-model. Since this has not yet been done to date, we will use a result of Orlov [41], who established mathematically a correspondence between the 'category of singularities' D_{Sg}, and the category of matrix factorisations (with the equivalence relations mentioned above). The category of singularities D_{Sg} is a certain quotient of the derived category of coherent sheaves that depends only on the singularity and would be empty on a smooth manifold.

Orlov's equivalence was formulated for the case of the un-orbifolded Landau-Ginzburg model. Given a matrix factorisation $W = EJ$ one interprets the two factors as maps between projective modules over the polynomial ring $\mathbb{C}[x_i]$ — in our case these modules are simply $\mathbb{C}[x_i]^{\oplus n}$ for a factorisation in terms of $n \times n$ matrices E, J. One then associates to a factorisation the object coker J, which naturally lives on $W = 0$. This assignment associates to a single transposition brane in an unorbifolded two variable model the set $x_1 - \eta x_2 = 0$. It has been shown [32] that the geometric intersection numbers can be matched with the intersection numbers derived from matrix factorisations (as well as with those obtained from permutation boundary states in conformal field theory). For the case of graded matrix factorisations in Landau-Ginzburg orbifolds, the idea is then that linear factorisations still describe the geometric object coker J for one choice of the representations γ_M; the D-branes corresponding to the other representations are images of that brane under the Landau-Ginzburg monodromy. For a number of examples this assignment has been verified for linear transposition and tensor product factorisations in [40, 32]. This was done by using alternative methods [42] to calculate the large volume charges corresponding to the branes at the Landau-Ginzburg point. In this paper, we will use these ideas to guess linear matrix factorisations corresponding to certain geometric D-branes.

3.2 Basic factorisations

The factorisations we shall mainly consider in this paper can be obtained as graded tensor products $Q_1 \odot Q_2$ [40, 12] of two simple classes of factorisations. The first construction involves a single factor theory of the form $W = x^h$, for which we can construct a factorisation as

$$Q(x) = \begin{pmatrix} 0 & x \\ x^{h-1} & 0 \end{pmatrix} . \tag{3.7}$$

The branes that correspond [40] to the tensor product of four such factorisations are the RS D-branes with $L = 0$ [43]. It follows from (3.6) that these branes do not couple to RR charges in the untwisted sector.

The other construction involves two factors of the form $W(x_1, x_2) = x_1^h + x_2^h$. Let η denote an h^{th} root of -1, then we have the factorisation [40]

$$Q_\eta = \begin{pmatrix} 0 & (x_1 - \eta x_2) \\ \prod_{\eta' \neq \eta}(x_1 - \eta' x_2) & 0 \end{pmatrix} . \tag{3.8}$$

It was shown in [32] (see also [33]) that the corresponding branes are permutation branes with $L = 0$ [44]. More generally, these factorisations can also be constructed for the case that h_1 and h_2 have a non-trivial common factor (but are not equal) [11]. The corresponding branes should then be generalised permutation branes similar to those considered in [45] in the case of group manifolds, [see [46] for a first attempt to formulate a related class of branes for coset theories].

If we tensor this permutation factorisation to two tensor factorisations, we get a transposition brane, denoted for example by (34). Once again, it only couples to charges in the twisted sectors. One can, of course, also consider tensoring with another permutation brane whenever the divisibility properties of the weights allow this. We will call the resulting branes double transposition branes and denote them by (12)(34), *etc.*

3.3 Deformations

For the following it will also be important to understand how the B-type D-branes behave under deformations of the complex structure. In particular, we will consider deformations of the Landau-Ginzburg superpotential by suitable quasihomogeneous polynomials of appropriate weight V, $W \mapsto \widehat{W}(\psi) = W + \psi V$, where ψ denotes the parameter of the deformation. If Q is a factorisation of W, then we ask whether there is $\widehat{Q}(\psi) = Q + f(\psi)\delta Q$ with $f(0) = 0$ such that $\widehat{Q}$ is a factorisation of $\widehat{W}$. If such a $\widehat{Q}(\psi)$ exists (at least in the neighbourhood of $\psi = 0$) we shall say that the D-brane can be extended for the deformation described by V.

3.3.1 Global deformations

There exist some classes of branes that can be extended for all possible deformations. In particular, this is the case for the tensor factorisations that correspond to RS branes. In order to see this we note that we can write any superpotential $\widehat{W}$ as

$$\widehat{W} = x_1\, F_1 + x_2\, F_2 + x_3\, F_3 + x_4\, F_4\,, \tag{3.9}$$

where the F are suitable polynomials. In fact, we have

$$F_i = \frac{w_i}{H}\,\frac{\partial \widehat{W}}{\partial x_i}\,. \tag{3.10}$$

We can thus define factorisations that are 4-fold tensor products of the factorisations $x_i F_i$. These factorisations are the deformations of the standard tensor factorisations. Indeed, as we approach the Gepner point, we have $F_i \to x_i^{h_i - 1}$, and these factorisations reduce to the tensor branes.

In a similar way, we can extend single transposition branes. For definiteness we assume that $w_3 = w_4$ and define

$$L_1 = x_1\,, \quad L_2 = x_2\,, \qquad L_3 = x_3 - \alpha x_4\,. \tag{3.11}$$

Inserting $L_1 = L_2 = L_3 = 0$ into the superpotential and imposing $\widehat{W} = 0$, one derives an equation of degree $k_3 + 2$ for α,

$$\widehat{W}(0, 0, \alpha, 1) = 0\,. \tag{3.12}$$

For each solution the Nullstellensatz then gives us a factorisation $\widehat{W} = L_1F_1 + L_2F_2 + L_3F_3$. At the Gepner point, the solutions for α are given by the $(k_3 + 2)$th roots of -1 and the factorisation reduces to the transposition brane, as required.

Of particular interest is the case where $w_3 = w_4 \neq 1$. In this case we are geometrically in the situation that the projective space acquires a singularity and the hypersurface intersects with it for generic complex structure deformations. The intersection points are then exactly given by $L_1 = L_2 = L_3 = 0$, where α solves (3.12).

We should note that in both cases, the factorisations that can be deformed do not couple to the charges in the untwisted sector (that are in turn in one-to-one correspondence to the polynomial deformation moduli). We shall see later on that this is indeed a necessary condition for the deformation to be possible.

3.3.2 Enhancement of the Picard lattice

As we have seen in section 2, the rank ρ of the Picard lattice is enhanced for hypersurfaces in weighted projective space. We would now like to understand this enhancement from the point of view of matrix factorisations.

Let us first discuss the part of the charge lattice that is inherited from the embedding space. For this, we consider the tensor product factorisations that correspond to the RS-branes. As we have just seen, these factorisations exist for arbitrary deformations of the superpotential. We expect on general grounds [47 – 50] that these branes carry precisely all the charges that can be obtained as pullbacks from the embedding space. If this is so, then it follows from the discussion in section 2 that their rank should be

$$\mathrm{rk}(\mathrm{tensor}) = 3 + \sum_{i<j} \Big(\gcd(w_i, w_j) - 1 \Big)\,. \tag{3.13}$$

Here, the 3 represents the D0, D4 and generic D2 charge, and the other contribution comes from the resolution of the singularities of the embedding weighted projective space. We have verified that this relation is indeed true for all 14 examples; the relevant rank is given in the penultimate column of table 1. This gives good support to the assertion that the tensor factorisations account precisely for the charges that can be understood in terms of the embedding projective space.

As we have seen, a $\mathbb{Z}_m$ singularity of the embedding space can lead to an enhancement of the rank of the Picard lattice of the hypersurface by a multiple of $m-1$ if the hypersurface intersects the exceptional locus in more than one point. For example, the rank of the Picard lattice of the example $\mathbb{P}_{(1,1,2,2)}[6]$ was shown to have 4 as a lower bound, where 3

holomorphic curves come from replacing the points $z_1 = z_2 = 0$, $z_3 - \eta z_4 = 0$, $\eta^3 = -1$ by $\mathbb{P}_1$'s. It is now natural to believe that these additional charges can be obtained as matrix factorisations of type (34), where η appears as the parameter in the (34) part of the factorisation.

In order to check this claim we have verified that for each η, the rank of the charge lattice of the tensor and (34) factorisations is bigger by one than that of the tensor factorisations. Furthermore, if we consider two different (34) factorisations with different η, the rank is increased by 2, but considering all three different values does not increase the rank any further (since the symmetric combination of the three η values is already part of the tensor charges). Furthermore, as we have just seen, all of these factorisations can be defined for arbitrary complex structure deformations. This explains from a matrix factorisation point of view that for $\mathbb{P}_{(1,1,2,2)}[6]$ $\rho \geq 4$ at a generic point in the complex structure moduli space.

We have studied these phenomena also for the other examples. The rank of the charge lattice spanned by the (34) branes, where $\mathrm{lcm}(w_3, w_4) = m$ is (for fixed value of η) always by $m - 1$ bigger than the rank of the tensor product lattice. Furthermore, including all values of η we obtain the generic part of the Picard lattice that arises because the K3 surface is embedded in the weighted projective space under consideration. The rank of the charge lattice that is generated by these factorisations is given in the last column of table 1; this agrees always with what is expected based on the geometric analysis of section 2.

3.4 An infinitesimal analysis

In section 3.3 we considered special factorisations that could be globally deformed. We would now like to investigate under which conditions a given factorisation can at least be infinitesimally deformed. Given Q_0, we want to find a $Q(\psi)$ with $Q(\psi) \to Q_0$ for $\psi \to 0$ such that

$$Q(\psi)^2 = W + \psi V \tag{3.14}$$

at least for small ψ. We make the analytic ansatz [12]

$$Q(\psi) = \sum_n \psi^n Q_n \,, \tag{3.15}$$

and obtain to first order

$$\{Q_0, Q_1\} = \psi V \,. \tag{3.16}$$

As $V \cdot \mathbf{1}$ is Q_0-closed, this reduces to a cohomology problem: if V is not exact, then Q_0 is obstructed and cannot be continued. At higher order we obtain similar conditions: since

$$\{Q_0, Q_n\} = -\sum_{k=1}^{n-1} Q_k Q_{n-k} \tag{3.17}$$

the right hand side must be Q_0-closed as well. In principle, obstructions may occur at higher orders too, but we have not found any examples where higher order obstructions were important.

The above ansatz (3.15) implies that the deformation is analytic, but it is conceivable that non-analytic deformations could exist. In particular, holomorphicity implies that there is only one smooth family of brane deformations, but physically there are certainly situations where more than a single deformation could compensate for a given bulk perturbation. In such cases we would expect non-analytic behavior of $Q(\psi)$. We can make a more general ansatz by making ψ an analytic function of a parameter ϕ:

$$\left(\sum_n \phi^n Q_n\right)^2 = W + \psi(\phi)V = W + \sum_n c_n \phi^n \cdot V\,. \tag{3.18}$$

However, this new ansatz is in fact only more general than (3.15) if the cohomology of Q_0 is non-trivial. (Physically, this corresponds to Q_0 having fermions in its self-spectrum.)

While we cannot solve the obstruction problem in general, we can at least give a necessary condition for the analytic deformation to be unobstructed.

3.4.1 A necessary condition for unobstructed deformations

We can make a general statement about the conditions that allow a brane to be continued: if Q can be continued analytically under the deformation V, then Q is not charged with respect to the corresponding RR ground state ϕ.

To prove this we start out with (3.6) in the untwisted sector. First of all, it is clear that if r is odd, the charge is always zero. We can thus assume that there is an even number of factors (as is always the case for the K3 examples). The charge is given by

$$\begin{aligned}\mathrm{ch}(Q) &= \oint dx_1 \dots dx_r \frac{V\mathrm{Str}[\partial_1 Q \dots \partial_r Q]}{\partial_1 W \dots \partial_r W} = \oint dx_1 \dots dx_r \frac{\mathrm{Str}[V\mathbf{1}\partial_1 Q \dots \partial_r Q]}{\partial_1 W \dots \partial_r W} \\ &= \oint dx_1 \dots dx_r \frac{\mathrm{Str}[\{Q,A\}\partial_1 Q \dots \partial_r Q]}{\partial_1 W \dots \partial_r W}\,,\end{aligned} \tag{3.19}$$

where we have used that V must be exact if Q can be continued. Consider now terms of the form $Q\partial Q$. Since $Q\partial Q = \partial(Q^2) - \partial Q Q$ and $Q^2 = W\mathbf{1}$,

$$\oint dx_1 \dots dx_r \frac{\mathrm{Str}[\partial_1 Q \dots \partial_k(Q^2) \dots \partial_r Q]}{\partial_1 W \dots \partial_r W} = \oint dx_1 \dots dx_r \frac{\partial_i W \mathrm{Str}[\partial_1 Q \dots\dots \partial_r Q]}{\partial_1 W \dots \partial_i W \dots \partial_r W}$$

and $\partial_i W$ cancels. At all Gepner points, $W = x_1^{h_1} + \ldots + x_r^{h_r}$, so x_i only appears in the numerator. The residue integral $\oint dx_i$ it thus zero. (This argument works also if W is not of the particular form given above, see [6].) This calculation shows that ∂Q and Q anticommute in the supertrace. Pulling Q through all the factors and using (anti-)cyclicity of the supertrace, we see that QA cancels with AQ and $\mathrm{ch}(Q)$ is thus zero.

This proof works for the twisted sector as well. In this case it suffices to realise that Q_n commutes with γ_M^n. This follows from the fact that

$$\gamma_M^n\, Q_n(\omega^{nw_i} x_i) = Q_n(x_i)\, \gamma_M^n\,. \tag{3.20}$$

But according to the definition of Q_n, only those x_i appear for which $\omega^{nw_i} = 1$. It is also clear that if we insert any fermionic boundary operator F such that $\{Q, F\} = 0$, the charge

remains zero. If there is an odd number of factors now, this is trivial, otherwise F just provides the additional sign that makes the trace disappear.

We note in passing that the globally deformed branes that we discussed in section 3.3.1 do indeed satisfy this condition.

3.4.2 A counterexample

In the previous subsection we have seen that a necessary condition for the brane Q not to be obstructed under the deformation V is that Q is not charged under V. One may wonder whether this condition is also sufficient, but this is not true. As an explicit counterexample consider the deformation of the (12)(34) brane of the quartic under the deformation $x_1^2 x_3^2$. If we choose the two values of η to be η_1 and η_2, $Q^{(1)} = Q_{\eta_1}(x_1, x_2) \odot Q_{\eta_2}(x_3, x_4)$, then their charge is

$$\mathrm{ch}(Q^{(1)})(x_1^2 x_3^2) = \frac{\eta_1^3 \eta_2^3}{16}\,. \tag{3.21}$$

Now define $Q^{(2)} = Q_{\eta_1}(x_1, x_2) \odot Q_{-\eta_2}(x_3, x_4)$, and consider the superposition Q of these two factorisations

$$Q = \begin{pmatrix} 0 & 0 & J_1 & 0 \\ 0 & 0 & 0 & J_2 \\ E_1 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 \end{pmatrix}\,. \tag{3.22}$$

Because of (3.21), this factorisation is then uncharged under $x_1^2 x_3^2$. Nevertheless it cannot be analytically deformed. If it could, we would have to find a matrix

$$X = \begin{pmatrix} 0 & 0 & A & B \\ 0 & 0 & C & D \\ E & F & 0 & 0 \\ G & H & 0 & 0 \end{pmatrix} \tag{3.23}$$

consisting of polynomial block matrices $A, B, \ldots, H$ such that

$$\{Q, X\} = x_1^2 x_3^2\, \mathbf{1}\,. \tag{3.24}$$

This yields eight (matrix) equations. The first and the fifth one are

$$\begin{aligned} J_1 E + A E_1 &= x_1^2 x_3^2 \mathbf{1} \\ E_1 A + E J_1 &= x_1^2 x_3^2 \mathbf{1}\,. \end{aligned} \tag{3.25}$$

These are, however, the very equations we find if we want to deform $Q^{(1)}$ itself. On the other hand, we know that $Q^{(1)}$ is charged under $x_1^2 x_3^2$ and thus not analytically deformable, so (3.25) has no solution. This shows that Q is not analytically deformable either.

3.5 Quartics on the orbifold line

As an interesting application of the above techniques we now want to study the line of 'very attractive' quartics (2.5) [13] from a matrix factorisation perspective. First we need to collect some information regarding the RR charges.

It is easy to see that there are 21 monomial deformations of the quartic superpotential that have integer $U(1)$ charge in the closed string theory. These correspond to 21 RR ground states in the untwisted sector. 19 of these monomials are of charge 1 — the corresponding RR ground states have charge 0 and can couple to both A-type and B-type branes. The remaining two integer charge monomials 1 and $x_1^2x_2^2x_3^2x_4^2$ correspond to RR ground states which have $q_L = q_R \neq 0$ and hence can couple only to A-type branes. Furthermore, each of the twisted sectors gives rise to one RR ground state each that can couple to B-type branes.

Our first task is to find a set of matrix factorisations that span the full B-type charge lattice. At the Gepner point, such a set is given by the double transposition branes (12)(34), (13)(24), (14)(23), which span a charge lattice of rank 22, accounting for the D0, D4 and 20 D2 branes. This in particular verifies that the Picard lattice at the Gepner point has maximal rank (namely 20). In the following, we want to analyse the deformations of these factorisations along the orbifold line.

3.5.1 Deformations of the Gepner point

It follows from our general discussion above that the D0-brane factorisation (34) and the tensor factorisation can be extended over the full complex structure moduli space, and therefore in particular, also along the orbifold line. These branes only couple to the three twisted RR charges, and thus account for the three generic RR-charges (that correspond to the D0, the D4, and the one D2-brane).

Next we observe that the (12)(34) factorisations can also be continued to arbitrary points on the orbifold line. To see this, we make the ansatz

$$L_1 = x_2 - \alpha_1 x_1\,, \qquad L_2 = x_3 - \alpha_2 x_4\,, \tag{3.26}$$

and insert $L_1 = L_2 = 0$ into the superpotential. To obtain a factorisation $W = L_1F_1 + L_2F_2$ from this ansatz, we require that the superpotential vanishes on the locus $L_1 = L_2 = 0$. In the case at hand this yields the following condition on the parameters

$$1 + \alpha_1^4 + a\alpha_1^2 = 0\,, \qquad 1 + \alpha_2^4 + b\alpha_2^2 = 0\,,$$

which is solvable for α_i for any value of a, b. In particular, this means that we have found a deformed (12)(34) factorisation for any value of a, b.

By the same argument we also see that the (13)(24) branes with

$$x_1 - \eta_1 x_3 = 0\,, \qquad x_2 - \eta_2 x_4 = 0 \tag{3.27}$$

can be extended to those a and b that satisfy $a\eta_1^2\eta_2^2 + b = 0$. For general parameters a, b however, (13)(24) and (14)(23) are obstructed, as follows immediately from the fact that they are charged under the corresponding deformations. On the other hand, it is possible to construct a factorisation of the deformed superpotential by writing it as

$$W(x) = \underbrace{\left(x_1^2 + \frac{a}{2}x_2^2\right)^2 + \left(x_3^2 + \frac{b}{2}x_4^2\right)^2}_{h=2} + \underbrace{\left(1 - \frac{a^2}{4}\right)x_2^4 + \left(1 - \frac{b^2}{4}\right)x_4^4}_{h=4}\,. \tag{3.28}$$

We can then consider the tensor product of the permutation factorisation of the first two and the last two terms. For $a \to 0, b \to 0$, these factorisations reduce to tensor products of permutation factorisations in x_1^2, x_3^2 and x_2, x_4 respectively. We can similarly combine x_1^2, x_4^2 and x_2, x_3, *etc.*; there are four different constructions of this type, and each accounts for two different charges. Together with the (12)(34) and the RS branes, they then span a charge lattice of rank 20.

Thus we have found a set of factorisations that can be deformed along the whole orbifold line and whose charges generate a sublattice of rank 20. As follows from the analysis of section 3.4.1, all of these factorisations are uncharged under $x_1^2 x_2^2$ and $x_3^2 x_4^2$, as is indeed also readily verified. The lattice of B-type D-branes that are uncharged under these two charges has rank 20, and thus the above constructions account already for all of it.

On the other hand, it also follows from our analysis of section 3.4.1, that *any* factorisation that is charged with respect to $x_1^2 x_2^2$ or $x_3^2 x_4^2$ cannot be analytically deformed. Furthermore, the (13)(24) brane for example does not have a fermion in its self-spectrum, and thus also the non-analytic solution of the type (3.18) cannot exist. This seems to predict that the corresponding D-brane in the orbifold theory should also be obstructed. We shall explain in detail in the next section that this is indeed so.

4. The conformal field theory description

In order to understand in more detail how the geometric and the matrix factorisation point of view fit together it is useful to study the quartic K3 surface (and its orbifold line) directly in conformal field theory. The conformal field theory we are interested in has two equivalent descriptions: it can be described as the Gepner model corresponding to the four-fold tensor product of four $N = 2$ minimal models with $k = 2$; on the other hand, the theory is also equivalent to the $\mathbb{Z}_4$ orbifold of a T^4-torus. The equivalence involves in fact mirror symmetry. In the following we shall first explain briefly the relevant Gepner model construction, and then describe in more detail the torus orbifold realisation and the correspondence between the two descriptions. Finally we shall describe some of the D-branes from both points of view, and explain how they deform under the Kaehler deformations of the orbifold theory.

4.1 The Gepner model

The Gepner description is standard [51], so we shall be fairly brief in the following. (A more comprehensive introduction to Gepner models can be found in [52]; our conventions are explained in more detail for example in [43, 32].)

The Gepner model of interest is the $\mathbb{Z}_4$-orbifold of the four-fold tensor product of $k = 2$ minimal models (each having $c = 3/2$, so that the total central charge is $c_{tot} = 6$). As usual we label the representations of the bosonic subalgebra of the $N = 2$ superconformal algebra by triples (l, m, s) of integers, where l takes the values $l = 0, 1, 2$, and m and s are defined modulo 8 and 4, respectively. The three integers have to obey $l + m + s = 0$

mod 2. Furthermore there is an identification

$$(l, m, s) \sim (2 - l, m + 4, s + 2)\,. \tag{4.1}$$

The conformal weight h and the U(1)-charge q of the highest weight state in the representation (l, m, s) are given by

$$h(l, m, s) = \frac{l(l+2) - m^2}{16} + \frac{s^2}{8} \mod \mathbb{Z}, \qquad q(l, m, s) = \frac{s}{2} - \frac{m}{4} \mod 2\mathbb{Z}. \tag{4.2}$$

Representations with s even belong to the Neveu-Schwarz sector, while those with s odd belong to the Ramond sector.
The space of states of the full theory is of the form

$$\bigotimes_{i=1}^{4} \mathcal{H}_{(l_i, m_i+n, s_i)} \otimes \bar{\mathcal{H}}_{(l_i, m_i-n, \bar{s}_i)}\,, \tag{4.3}$$

where $n = 0, 1, 2, 3$ denotes the twisted sector, and s_i (and $\bar{s}_i$) are all either even (NS) or all odd (R). The labels m_i are subject to the integrality condition

$$\sum_{i=1}^{4} \frac{m_i}{4} \in \mathbb{Z}\,. \tag{4.4}$$

Finally, we may impose the type 0B GSO-projection which requires that

$$\sum_{i=1}^{4} \left(\frac{s_i}{2} + \frac{\bar{s}_i}{2}\right) \in 2\mathbb{Z}\,. \tag{4.5}$$

Of particular importance are the RR ground states of this theory. Ramond ground states are characterised by the property that their conformal weight h equals $c/24$. One can easily show that the ground state of the sector (l, m, s) is a Ramond ground state if it is of the form $(l, l + 1, 1)$ or $(l, -l - 1, -1)$. The above Gepner model possesses 24 RR ground states; in each of the three twisted sectors ($n = 1, 2, 3$) there is one RR ground state which is the state

$$(n - 1, n, 1)^{\otimes 4} \otimes \overline{(n - 1, -n, -1)}^{\otimes 4}\,. \tag{4.6}$$

The remaining 21 RR ground states come from the untwisted $n = 0$ sector; if we associate to the R ground state representations

$$(0, 1, 1) \leftrightarrow 1\,, \qquad (1, 2, 1)_i \leftrightarrow x_i\,, \qquad (2, 3, 1)_i \leftrightarrow x_i^2\,, \tag{4.7}$$

where the index i refers to the ith factor, then we have the state 1, $x_1^2 x_2^2 x_3^3 x_4^2$, as well as the 19 monomials in x_i that are of degree 4.

4.2 The torus orbifold

The torus in question is simply the orthogonal product of four circles, which initially all have the self-dual radius and vanishing B-field. For the following it is convenient to write this 4-torus as $T^4 = T^2 \times T^2$. The $\mathbb{Z}_4$ orbifold acts by a counterclockwise rotation by 90 degrees in the first T^2, and by a clockwise rotation by 90 degrees in the second. We denote the four real directions by y^i with $i = 1, 2, 3, 4$, and introduce complex coordinates in the

usual way: $z^1 = y^1 + iy^2$ and $z^2 = y^3 + iy^4$. The $\mathbb{Z}_4$ action is then

$$g : z^1 \mapsto e^{\frac{2\pi i}{4}} z^1 , \qquad z^2 \mapsto e^{-\frac{2\pi i}{4}} z^2 . \tag{4.8}$$

We denote the fermionic fields by χ^i and $\tilde{\chi}^i$, where $i = 1, 2, 3, 4$. The corresponding complex fields are then

$$\psi^1 = \frac{1}{\sqrt{2}} \left(\chi^1 + i\chi^2\right) \quad \bar{\psi}^1 = \frac{1}{\sqrt{2}} \left(\chi^1 - i\chi^2\right) \quad \psi^2 = \frac{1}{\sqrt{2}} \left(\chi^3 + i\chi^4\right) \quad \bar{\psi}^2 = \frac{1}{\sqrt{2}} \left(\chi^3 - i\chi^4\right) \tag{4.9}$$

with similar formulae for the right-moving fields, $\tilde{\psi}^1$, *etc.*

The momentum ground states of the torus are labelled by four momentum numbers n_i, $i = 1, 2, 3, 4$, and four winding numbers w_j, $j = 1, 2, 3, 4$. For the ith direction the left- and right-moving momenta are then

$$\left(p_L^i, p_R^i\right) = \left(\frac{n_i}{2R_i} + w_i R_i, \frac{n_i}{2R_i} - w_i R_i\right) , \tag{4.10}$$

where initially all $R_i = \frac{1}{\sqrt{2}}$ in our conventions. On the ground states, the $\mathbb{Z}_4$ action maps

$$(n_1, w_1, n_2, w_2, n_3, w_3, n_4, w_4) \mapsto (-n_2, -w_2, n_1, w_1, n_4, w_4, -n_3, -w_3) . \tag{4.11}$$

This symmetry requires only that $R_1 = R_2$ and $R_3 = R_4$, but neither needs to take the self-dual value. Thus there is a two (real)-dimensional space of deformations that preserve the orbifold symmetry. One should expect on general grounds that this is only part of a two (complex)-dimensional space of deformations, and this is indeed so. One easily sees that one can also switch on an arbitrary B-field in either of the two T^2: if we concentrate on the first T^2, then the momenta are of the form

$$\begin{aligned}\left(p_L^1, p_L^2 | p_R^1, p_R^2\right) = \Big(&\frac{n_1}{2R} + w_1 R + B w_2, \frac{n_2}{2R} + w_2 R - B w_1 \Big| \\ &\frac{n_1}{2R} - w_1 R + B w_2, \frac{n_2}{2R} - w_2 R - B w_1\Big) .\end{aligned} \tag{4.12}$$

It is then easy to see that the spectrum is invariant under the $\mathbb{Z}_4$-action

$$(p_L^1, p_R^1) \mapsto (p_L^2, p_R^2) \mapsto (-p_L^1, -p_R^1) \mapsto (-p_L^2, -p_R^2) . \tag{4.13}$$

In fact, this action still corresponds precisely to the action (on the first two coordinates) of (4.11).

In the following we shall mainly concentrate on the theory where the B-field vanishes and all the radii take the self-dual value $R_i = \frac{1}{\sqrt{2}}$; this is the theory that corresponds precisely to the Gepner model $(2)^4$. Unless mentioned otherwise this is what we shall call the torus orbifold in the following.

4.3 A partial dictionary

Before proceeding we shall match a few low-lying states in order to understand how the identification works. In the untwisted NS-NS sector of the torus orbifold, the lowest lying states is the vacuum with $h = \bar{h} = 0$, as well as four states of $h = \bar{h} = 1/4$. The latter

are the $\mathbb{Z}_4$-orbits of the states for which the only non-vanishing momentum and winding number is $n_1 = 1$, or $w_1 = 1$ or $n_3 = 1$ or $w_3 = 1$. In the first (g) and third (g^3) twisted sector of the torus orbifold, there are 4 fixed points each, and each of them has (in the NS-NS sector) ground state energy $h = \bar{h} = 1/4$. (There are 10 $\mathbb{Z}_4$-orbifold invariant $\mathbb{Z}_2$-fixed points in the second (g^2) twisted sector, but their ground state energy is higher.) In total the torus orbifold therefore has 12 NS-NS states with $h = \bar{h} = 1/4$.

In the $N = 2$ orbifold (the Gepner model), all of these states appear in the untwisted ($n = 0$) sector; the vacuum obviously corresponds to the ground state of the trivial representations, and the states with $h = \bar{h} = 1/4$ are the ground states in the sectors

$$(1, \pm 1, 0) \otimes (1, \mp 1, 0) \otimes (0, 0, 0) \otimes (0, 0, 0)\,, \tag{4.14}$$

where the two non-trivial representations may appear in any two of the four factors (there are six different possibilities), and the two signs are correlated. [In the above we have only written the left-moving representations; since $n = 0$ the right-moving representations are simply equal.]

4.3.1 RR ground states

It is also instructive to understand how the 24 RR ground states of the $N = 2$ Gepner model that were described at the end of the previous section appear in the torus orbifold. In the untwisted sector of the torus orbifold we have eight fermionic zero modes, namely χ_0^i and $\tilde{\chi}_0^i$, or the corresponding complex modes defined by (4.9). We combine them into creation and annihilation operators by defining

$$\begin{aligned}
\psi_1^\pm &= \frac{1}{\sqrt{2}}\left(\psi_0^1 \pm i\tilde{\psi}_0^1\right) = \frac{1}{2}\left(\chi_0^1 + i\chi_0^2\right) \pm i\frac{1}{2}\left(\tilde{\chi}_0^1 + i\tilde{\chi}_0^2\right) \\
\bar{\psi}_1^\pm &= \frac{1}{\sqrt{2}}\left(\bar{\psi}_0^1 \pm i\tilde{\bar{\psi}}_0^1\right) = \frac{1}{2}\left(\chi_0^1 - i\chi_0^2\right) \pm i\frac{1}{2}\left(\tilde{\chi}_0^1 - i\tilde{\chi}_0^2\right) \\
\psi_2^\pm &= \frac{1}{\sqrt{2}}\left(\psi_0^2 \pm i\tilde{\psi}_0^2\right) = \frac{1}{2}\left(\chi_0^3 + i\chi_0^4\right) \pm i\frac{1}{2}\left(\tilde{\chi}_0^3 + i\tilde{\chi}_0^4\right) \\
\bar{\psi}_2^\pm &= \frac{1}{\sqrt{2}}\left(\bar{\psi}_0^2 \pm i\tilde{\bar{\psi}}_0^2\right) = \frac{1}{2}\left(\chi_0^3 - i\chi_0^4\right) \pm i\frac{1}{2}\left(\tilde{\chi}_0^3 - i\tilde{\chi}_0^4\right)\,.
\end{aligned}$$

We define $|0\rangle_{RR}$ to be the state that is annihilated by the $-$ modes, *i.e.*

$$\psi_j^- |0\rangle_{RR} = \bar{\psi}_j^- |0\rangle_{RR} = 0\,, \qquad j = 1, 2\,. \tag{4.15}$$

The space of RR ground states is thus generated by the action of the $+$-modes from this state. Since there are four creation operators, the space of RR ground states (before orbifold projection) is 16-dimensional.

The state $|0\rangle_{RR}$ can be taken to be invariant under the orbifold action, while the ψ-modes transform as

$$\begin{aligned}
g\,\psi_1^\pm g^{-1} &= e^{\frac{2\pi i}{4}}\,\psi_1^\pm\,, \qquad & g\,\bar{\psi}_1^\pm g^{-1} &= e^{-\frac{2\pi i}{4}}\,\bar{\psi}_1^\pm\,, \\
g\,\psi_2^\pm g^{-1} &= e^{-\frac{2\pi i}{4}}\,\psi_2^\pm\,, \qquad & g\,\bar{\psi}_2^\pm g^{-1} &= e^{\frac{2\pi i}{4}}\,\bar{\psi}_2^\pm\,.
\end{aligned} \tag{4.16}$$

Of the 16 RR ground states, there are therefore six states that are invariant under the $\mathbb{Z}_4$ orbifold action: in addition to $|0\rangle_{RR}$ and $\psi_1^+\bar{\psi}_1^+\psi_2^+\bar{\psi}_2^+|0\rangle_{RR}$ they are

$$\psi_1^+\bar{\psi}_1^+|0\rangle_{RR}\,, \qquad \psi_1^+\psi_2^+|0\rangle_{RR}\,, \qquad \bar{\psi}_1^+\bar{\psi}_2^+|0\rangle_{RR}\,, \qquad \psi_2^+\bar{\psi}_2^+|0\rangle_{RR}\,. \tag{4.17}$$

Two linear combinations of these six states can only couple to B-type branes, two can only couple to A-type branes, while the remaining two can couple to either. (A more explicit analysis of the $N = 2$ charges of these states is spelled out in appendix A.2.) The relevant A-type branes are the mirror of the D0- and the D4-brane, as well as of two B-type D2-branes that contribute to the Picard lattice.

The remaining 18 RR charges arise from the twisted sector. In order to describe the twisted sector states it is useful to think of the $\mathbb{Z}_4$ orbifold in two steps as the $\mathbb{Z}_2$ orbifold of a $\mathbb{Z}_2$ orbifold. The first $\mathbb{Z}_2$ orbifold inverts all four torus coordinates, while the second $\mathbb{Z}_2'$ orbifold acts as a rotation by 90 degrees in the two tori (clockwise in the first, and anti-clockwise in the second). The first $\mathbb{Z}_2$ orbifold has, as usual, 16 fixed points at $\frac{1}{2}(y_1, y_2, y_3, y_4)$, where each y_i is either 0 or 1. Only four of these fixed points are invariant under the full $\mathbb{Z}_4$ orbifold action, namely $(0,0,0,0)$, $(1,1,0,0)$, $(0,0,1,1)$ and $(1,1,1,1)$. Thus we have four RR ground states in each of the g, g^2 and g^3-twisted sectors, giving together 12 RR ground states. This also fits together with the geometric description of the orbifold: at each $\mathbb{Z}_4$ singular point the geometry looks locally like $\mathbb{C}_2/\mathbb{Z}_4$, which is a singularity of type A_3. Its resolution introduces 3 exceptional divisors whose intersection pattern is determined by the corresponding Cartan matrix.

The other 12 $\mathbb{Z}_2$-fixed points form orbits of length 2 under the additional $\mathbb{Z}_2'$ action, leading to 6 $\mathbb{Z}_2$ fixed points of the full $\mathbb{Z}_4 = \mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold. (Each of these introduces a single exceptional divisor.) They therefore only contribute 6 states to the g^2 twisted sector. In total we therefore have 18 twisted RR ground states. All of them correspond to 2-cycles that are part of the Picard lattice. Together with the two charges that appear in the untwisted sector we thus see that the rank of the Picard lattice of the mirror is indeed maximal, $\rho = 20$.

The counting of the RR charges is obviously valid at any point on the orbifold line. One might wonder which of the 2-cycles are special, so that they cannot be deformed easily in the matrix factorisation picture. In fact, one would expect that the 18 2-cycles from the blow-up contribute in a straight-forward manner at any point along the orbifold line. Indeed, this is in analogy to our discussion of hypersurfaces in weighted projective space where the 2-cycles coming from the resolution contribute everywhere in moduli space. On the other hand, the remaining 2-cycles that come from the torus do depend more critically on the radii. As we shall see, this expectation will indeed be borne out.

4.3.2 Quantum symmetries

Finally, it is very instructive to identify the quantum symmetries of the orbifolds on both sides. The quantum symmetry of the torus orbifold acts on the Gepner model as

$$e^{\frac{i\pi}{2}(m_1+m_2-s_1-s_2)}\,. \tag{4.18}$$

In fact, this identification can be read off from the geometric point of view that will be described below in section 4.4. The states from the g^0 (untwisted) sector of the orbifold theory thus correspond to the polynomials

$$g^0 \longleftrightarrow 1\,, \quad x_1^2x_2^2\,, \quad x_3^2x_4^2\,, \quad x_1^2x_2^2x_3^2x_4^2\,, \tag{4.19}$$

and to the $(n = 1)$ and $(n = 3)$ twisted RR ground states. The four RR ground states of the g-twisted sector are

$$g^1 \longleftrightarrow x_1x_3x_4^2\,, \quad x_2x_3x_4^2\,, \quad x_1x_4x_3^2\,, \quad x_2x_4x_3^2\,, \tag{4.20}$$

while the corresponding statement for the g^3-twisted sector is

$$g^3 \longleftrightarrow x_1^2x_2x_3\,, \quad x_1^2x_2x_4\,, \quad x_2^2x_1x_3\,, \quad x_2^2x_1x_4\,. \tag{4.21}$$

All other RR ground states come from the g^2 twisted sector.

Conversely, we can also identify the quantum symmetry of the $N = 2$ orbifold that appears in the construction of the Gepner model, on the torus side: as will become clear from the detailed analysis of the appendix A.2 it seems to be given by the $\mathbb{Z}_4$ rotation by 90 degrees that only acts on the first T^2, but leaves the second T^2 invariant. The three RR ground states in the Gepner model that appear in the twisted sectors $(n = 1, 2, 3)$ then correspond to the three RR ground states of the torus orbifold that have eigenvalues $e^{\frac{2\pi in}{4}}$ under this 90 degree rotation. The state with $n = 2$ corresponds to a specific $\mathbb{Z}_4$-invariant combination of $\mathbb{Z}_2$-fixed points from the g^2 sector of the torus orbifold, while the $n = 1$ and $n = 3$ states arise from the untwisted sector of the torus orbifold. In fact one easily sees that

$$\begin{aligned} &(n=1) \qquad (0,1,1)^{\otimes 4} \otimes \overline{(0,-1,1)}^{\otimes 4} \longleftrightarrow \psi_1^+\psi_2^+|0\rangle_{RR} \\ &(n=3) \qquad (2,3,1)^{\otimes 4} \otimes \overline{(2,-3,1)}^{\otimes 4} \longleftrightarrow \bar\psi_1^+\bar\psi_2^+|0\rangle_{RR}\,. \end{aligned} \tag{4.22}$$

This identification will prove very useful below.

4.4 The Inose point of view

The equivalence of the $\mathbb{Z}_4$ orbifold line with certain perturbations of the Gepner quartic can also be understood [13] as an extension of a purely geometric result due to Inose [57]. This point of view also ties in nicely with the identification of the the D-branes of the two theories.

Inose discovered that the K3 surface obtained as a resolution of the toroidal $\mathbb{Z}_2$ orbifold is equivalent to a geometric $\mathbb{Z}_2$ orbifold of the quartic K3 at the Gepner point. As before, the $\mathbb{Z}_2$ action on the torus is given by inversion of all 4 coordinates, whereas the $\mathbb{Z}_2$ action on the hypersurface acts as

$$\sigma : (x_1, x_2, x_3, x_4) \to (-x_1, -x_2, x_3, x_4)\,. \tag{4.23}$$

This orbifold action has 8 fixed points $(x_1, x_2, 0, 0)$ with $x_1^4 + x_2^4 = 0$, and $(0, 0, x_3, x_4)$ with $x_3^4 + x_4^4 = 0$, introducing 8 exceptional $\mathbb{P}_1$'s. The lines

$$x_1 - \eta x_2 = 0\,, \quad x_3 - \eta x_4 = 0\,, \tag{4.24}$$

of which one would expect to have a Landau-Ginzburg description in terms of the corresponding matrix factorisations (12)(34), are invariant under the orbifold action. According to Inose, the 16 lines (4.24) correspond to the 16 $\mathbb{P}_1$'s required to resolve the $\mathbb{Z}_2$ singularities of $T^4/\mathbb{Z}_2$.

To apply Inose's result to the relation between the toroidal $\mathbb{Z}_4$ orbifold and the quartic hypersurface we remember that in conformal field theory any abelian orbifold possesses a quantum symmetry by means of which one can undo the orbifold. This quantum symmetry acts on the twisted sectors by phase multiplication, and if we orbifold the orbifold theory by this quantum symmetry we reobtain the original theory. In the case at hand, one would like to divide out the orbifolded quartic by its quantum symmetry to re-obtain the quartic. [13] identifies this quantum symmetry on the toroidal side: it is precisely the $\mathbb{Z}_2'$ that enhances the $\mathbb{Z}_2$ given by coordinate inversion to the $\mathbb{Z}_4$ action (4.8).

This can now help us to understand to which twisted sectors (of the torus theory) the branes of the $(2)^4$ model should couple. Let us first note that σ acts on the states in the Gepner model as

$$\sigma : \otimes(l_i, m_i, s_i) \to (-1)^{l_1+l_2} \otimes (l_i, m_i, s_i)\,. \tag{4.25}$$

As mentioned before, the lines (4.24) on the quartic hypersurface are invariant under the $\mathbb{Z}_2$ action induced by σ. This means that also the corresponding matrix factorisations are invariant. In conformal field theory language, the boundary states corresponding to the (12)(34) transposition branes are thus invariant under the σ-orbifold operation, and need to be 'resolved' by adding a contribution from the twisted sector, with a choice of sign reflecting the freedom to pick a representation on the Chan-Paton labels.

Going back to the covering theory by dividing out by the quantum symmetry, these resolved boundary states then form orbits under the quantum symmetry and need not be resolved again. On the torus side, the same should happen, and the corresponding boundary states should therefore only couple to the $\mathbb{Z}_2$ fixed points points, but not to the g or g^3 twisted sector of the torus orbifold. In fact, this is in agreement with the above identification (4.20) and (4.21) since it follows from (3.6) that the (12)(34) factorisations are not charged under these monomials.

On the other hand the (13)(24) matrix factorisations are not invariant under σ, and therefore form orbits under σ-orbifold action. In turn, they therefore need to be resolved under the quantum symmetry orbifold. On the torus side, we should therefore expect that the corresponding boundary states do couple to the g and g^3 twisted sectors; again this is in agreement with the identifications (4.20), (4.21) and the charge formula (3.6).

4.5 Some simple D-branes

Having understood at least in parts the dictionary between the Gepner model and the torus orbifold description, we now want to explain the torus description of certain classes of branes in Gepner models. In particular, we want to study the tensor product (RS) branes [43] and the permutation branes [44] whose matrix factorisation description was explained in [40, 53, 32, 54, 33]. For a related theory, namely $T^2/\mathbb{Z}_4$, this analysis was recently performed in [55] (see also [56]).

4.5.1 The tensor branes

The simplest D-branes of the Gepner model are the RS D-branes that correspond to tensor products of rank 1 factorisations of the separate factors. In analogy to the correspondence between pure D6 branes and their images under the Gepner monodromy and the $L_i = 0$ RS states one expects that these states correspond geometrically to D4 branes wrapped on the quartic hypersurface. Via mirror symmetry, they are mapped to D2 branes on the torus orbifold. To be more precise, these D2 should have one Neumann and one Dirichlet direction in each torus. In fact, one easily sees that the RS-branes do not couple to the twelve NS-NS states of (4.14); the RS-branes therefore cannot be D0- or D4-branes and thus must be D2-branes.

The RS D-branes only couple to the RR ground states in the twisted sectors ($n \neq 0$). Given the identification (4.22) as well as the explicit formulae for these boundary states (see for example [32] whose conventions we employ in the following) we thus know that the RS branes with $L_i = 0$ couple to

$$|\mathrm{RS}\rangle\!\rangle^0 \simeq \left(e^{-\frac{\pi i \hat{M}}{4}}\, \psi_1^+ \psi_2^+ + e^{\frac{\pi i \hat{M}}{4}}\, \bar{\psi}_1^+ \bar{\psi}_2^+\right) |0\rangle_{RR}\,. \tag{4.26}$$

Since $\hat{M}$ is even, these D-branes therefore only couple to differences and sums of these two torus states. With the identification of the previous section, it is furthermore clear that these are the only RR ground states of the untwisted sector of the torus orbifold to which these D-branes couple. (The other such states arise in the $n = 0$ sector of the $N = 2$ orbifold, to which the RS-branes do not couple.)

In order to determine their orientation we rewrite (4.26) in terms of the real coordinates. We find

$$\begin{aligned} \left(\psi_1^+ \psi_2^+ + \bar{\psi}_1^+ \bar{\psi}_2^+\right) |0\rangle_{RR} &= (\chi_1^+ \chi_3^+ - \chi_2^+ \chi_4^+)|0\rangle_{RR} \\ i\left(\psi_1^+ \psi_2^+ - \bar{\psi}_1^+ \bar{\psi}_2^+\right) |0\rangle_{RR} &= (\chi_2^+ \chi_3^+ + \chi_1^+ \chi_4^+)|0\rangle_{RR}\,, \end{aligned} \tag{4.27}$$

where $\chi_i^+ = \chi_i + i\tilde{\chi}_i$. In the former case (which corresponds to $\hat{M} = 0 \bmod 4$), the branes are the superposition of branes with Neumann directions along y^1 and y^3, and branes with Neumann directions along y^2 and y^4. In the latter case the relevant D2-branes have Neumann directions along y^2 and y^3, and Neumann directions along y^1 and y^4. These superpositions are then $\mathbb{Z}_4$-orbifold invariant.

4.5.2 The transposition branes

The next simplest class of D-branes are the transposition branes corresponding to the permutation (ij), where $i \neq j$. These branes couple to the same RR ground states as the tensor branes. Their coupling is however different: taking into account the subtle factor in the relative overlaps to the tensor branes (see eq. (5.7) of [32]), we find that the branes (again with $L_i = 0$) couple instead of (4.26) to

$$|(ij)\rangle\!\rangle^0 \simeq \frac{1}{\sqrt{2}} \left(e^{-\frac{\pi i(\hat{M}+1)}{4}}\, \psi_1^+ \psi_2^+ + e^{\frac{\pi i(\hat{M}+1)}{4}}\, \bar{\psi}_1^+ \bar{\psi}_2^+\right) |0\rangle_{RR}\,. \tag{4.28}$$

Since $\hat{M}$ is even, these branes therefore couple to different linear combinations; in terms of the real coordinates, the ground state is proportional to

$$\begin{aligned}\left(\psi_1^+\psi_2^+ + i\bar{\psi}_1^+\bar{\psi}_2^+\right)|0\rangle_{RR} &= (1+i)\left[\chi_1^+(\chi_3^+ + \chi_4^+) + \chi_2^+(\chi_3^+ - \chi_4^+)\right]|0\rangle_{RR} \\ \left(\psi_1^+\psi_2^+ - i\bar{\psi}_1^+\bar{\psi}_2^+\right)|0\rangle_{RR} &= (1-i)\left[\chi_1^+(\chi_3^+ - \chi_4^+) - \chi_2^+(\chi_3^+ + \chi_4^+)\right]|0\rangle_{RR}\,.\end{aligned}$$

These branes are therefore superpositions of D2-branes that have Neumann directions along y^1 and $y^3 \pm y^4$, and branes with Neumann directions along y^2 and $y^3 \mp y^4$.

4.5.3 The double transposition branes

The last simple class of branes corresponds to the product of two transpositions, *i.e.* to the permutation $(ij)(kl)$ with i,j,k,l all mutually distinct. These branes also couple to untwisted ($n = 0$) RR ground states, and may therefore also couple to additional RR ground states of the untwisted (g^0) torus orbifold. As regards the two RR ground states coming from $n = 1$ and $n = 3$, their coupling is now (again for $L_i = 0$)

$$|(ij)(kl)\rangle\!\rangle^0 \sim \frac{1}{2}\left(e^{-\frac{\pi i(\hat{M}+2)}{4}}\,\psi_1^+\psi_2^+ + e^{\frac{\pi i(\hat{M}+2)}{4}}\,\bar{\psi}_1^+\bar{\psi}_2^+\right)|0\rangle_{RR}\,. \tag{4.29}$$

The more detailed interpretation however depends on which permutation is considered.

The case (12)(34): In this case it follows from the identification of (4.20) and (4.21) that the (12)(34) branes do *not* couple to any RR ground states of the first (g^1) or third (g^3) twisted sector of the orbifold. This implies that they cannot be $\mathbb{Z}_4$-fractional branes, and thus that they must correspond to the superpositions of at least two D2-branes. The orientation of the two D2-branes is then as described for the tensor branes in 4.4.1. [The tension of the (12)(34) branes is smaller by a factor of two than that of the tensor branes; this suggests that the latter are actually superpositions of four such branes, while the (12)(34) only involve two D2-branes.]

The cases (13)(24) and (14)(23): In either case, the identification of (4.20) and (4.21) now implies that these branes *do* couple to the first (g^1) and third (g^3) twisted sector of the orbifold. Thus they should correspond to 'fractional' branes. We also know that the $(ij)(kl)$ branes may couple to additional RR ground states in the untwisted sector of the torus orbifold. In fact, since the set of all (double) transposition branes account for all RR charges, at least some of the (13)(24) and (14)(23) branes *must* couple to these states. From the point of view of the orbifold description, the relevant RR ground states are identified in appendix A.2. This then suggests that the RR ground states of some of these boundary states are proportional to

$$\begin{aligned}|B1\rangle\!\rangle^0 &\simeq (1+\psi_1^+\psi_2^+)\,(1+\bar{\psi}_1^+\bar{\psi}_2^+)|0\rangle_{RR} \\ |B2\rangle\!\rangle^0 &\simeq (\psi_1^+ + \bar{\psi}_2^+)\,(\psi_2^+ + \bar{\psi}_1^+)|0\rangle_{RR}\,.\end{aligned}$$

These ground states satisfy the gluing conditions

$$\begin{aligned}
&(\chi_1 - i\tilde{\chi}_3)|B1\rangle\rangle^0 = 0 \qquad &&(\chi_2 + i\tilde{\chi}_4)|B1\rangle\rangle^0 = 0 \\
&(\chi_3 + i\tilde{\chi}_1)|B1\rangle\rangle^0 = 0 \qquad &&(\chi_4 - i\tilde{\chi}_2)|B1\rangle\rangle^0 = 0\,, \\
&(\chi_1 + i\tilde{\chi}_3)|B2\rangle\rangle^0 = 0 \qquad &&(\chi_2 - i\tilde{\chi}_4)|B2\rangle\rangle^0 = 0 \\
&(\chi_3 + i\tilde{\chi}_1)|B2\rangle\rangle^0 = 0 \qquad &&(\chi_4 - i\tilde{\chi}_2)|B2\rangle\rangle^0 = 0\,.
\end{aligned} \tag{4.30}$$

The corresponding D2-branes lie diagonally across the two T^2s and are by themselves $\mathbb{Z}_4$-invariant (as should be the case for $\mathbb{Z}_4$-fractional branes!). On the other hand, their area is twice that of one of the two D2-branes that appears in the description of the (12)(34) brane. This is then in accord with the fact that the tension of the (12)(34) brane agrees with that of the (13)(24) and the (14)(23) branes.

4.6 Deforming D-branes

So far we have (partially) identified the D-branes of the orbifold theory at the Gepner point with certain classes of matrix factorisations. In terms of the orbifold theory, it is now not difficult to describe how these D-branes behave as we vary the radii or the B-fields.

First of all, it is clear that nothing much of interest happens for the branes that correspond to the tensor factorisations (section 4.5.1), the single transposition factorisation (section 4.5.2), or the (12)(34) branes. In all of these cases the gluing conditions involve the two T^2's separately, and the structure of these D-branes is pretty insensitive to changes of the radii or the B-field.

The situation is however different for the $|B1\rangle\rangle$ and $|B2\rangle\rangle$ branes since they lie diagonally across the two T^2s. As we change the radii of the two T^2's, we generically change their ratio, which has a significant effect on the behaviour of these branes. A priori it is not clear how we should 'continue' these D-branes as we vary the closed string parameters, but there are at least two natural points of views that we can take.

According to the first point of view, we can simply insist on preserving the same gluing conditions (4.30) (as well as the corresponding gluing conditions for the bosons) as we vary the radii and the B-fields. By construction, the corresponding D-branes will then continue to couple to the relevant RR ground states, and will continue to satisfy the correct $N = 2$ gluing conditions. However, as is well known [14], the structure of the corresponding D-branes will depend dramatically on the precise ratio of the radii and the values of the B-fields. Consider for example the $\|B1\rangle\rangle$ brane that is characterised by the bosonic gluing conditions corresponding to (4.30)

$$\begin{aligned}
&(a^1_n - \tilde{a}^3_{-n})\|B1\rangle\rangle = 0\,, \qquad &&(a^2_n + \tilde{a}^4_{-n})\|B1\rangle\rangle = 0\,, \\
&(a^3_n + \tilde{a}^1_{-n})\|B1\rangle\rangle = 0\,, \qquad &&(a^4_n - \tilde{a}^2_{-n})\|B1\rangle\rangle = 0\,.
\end{aligned} \tag{4.31}$$

For the original theory for which $R_1 = R_2 = \frac{1}{\sqrt{2}}$ and $B_1 = B_2 = 0$, we have for example Ishibashi states on the momentum ground states for which the only non-vanishing momentum and winding numbers are $n_3 = w_1$ or $w_3 = -n_1$ or $n_4 = -w_2$ or $w_4 = n_2$. (Obviously the $\mathbb{Z}_4$-invariant boundary state will require that we sum over suitable such combinations of

Ishibashi states.) As we change the radii R_1 and R_2 or switch on the B-fields, the Ishibashi state whose ground state has only non-vanishing momentum and winding numbers equal to $n_3 = w_1 \neq 0$ say, does typically *not* satisfy the zero mode part of (4.31) any more. Thus the set of Ishibashi states that contribute will depend crucially on the closed string parameters. As a consequence, the same will be true for their tension, *etc.* This fact is also easy to understand geometrically: fixing the gluing conditions means that we fix the angles with which the D-brane is oriented in the 13- and 24-planes. As we change the ratio of the radii, the number of times the brane wraps around the torus in the 13 and 24 directions changes erratically.

In order to avoid this erratic behaviour, we can adopt the second point of view, namely that we should modify the gluing conditions as we change the radii or switch on a B-field. For example, for the case when we change the radii R_1 and R_2, we can consider

$$\begin{aligned}
\begin{pmatrix} a^1_n \\ a^3_n \end{pmatrix} &= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} \tilde{a}^1_{-n} \\ \tilde{a}^3_{-n} \end{pmatrix} , \\
\begin{pmatrix} a^2_n \\ a^4_n \end{pmatrix} &= \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} \tilde{a}^2_{-n} \\ \tilde{a}^4_{-n} \end{pmatrix} ,
\end{aligned} \tag{4.32}$$

where $2R_1R_2 = \tan\theta$. It is straightforward to check that the states with $n_3 = w_3 \neq 0$, *etc.* then satisfy the zero mode part of (4.32) for arbitrary values of θ (not just $\theta = \pi/4$). In order to understand the geometric meaning of these modified gluing conditions, we rewrite them in terms of complex coordinates. If we define $\alpha^{(1)}_n = a^1_n + i a^2_n$, *etc*, we find

$$\begin{aligned}
\alpha^{(1)}_n &= \cos 2\theta\, \tilde{\alpha}^{(1)}_{-n} + \sin 2\theta\, \tilde{\tilde{\alpha}}^{(2)}_{-n} & \qquad \bar{\alpha}^{(1)}_n &= \cos 2\theta\, \tilde{\tilde{\alpha}}^{(1)}_{-n} + \sin 2\theta\, \tilde{\alpha}^{(2)}_{-n} \\
\alpha^{(2)}_n &= \cos 2\theta\, \tilde{\alpha}^{(2)}_{-n} - \sin 2\theta\, \tilde{\tilde{\alpha}}^{(1)}_{-n} & \qquad \bar{\alpha}^{(2)}_n &= \cos 2\theta\, \tilde{\tilde{\alpha}}^{(2)}_{-n} - \sin 2\theta\, \tilde{\alpha}^{(1)}_{-n}
\end{aligned} \tag{4.33}$$

In order to preserve the usual world-sheet $N = 1$ algebra, the fermions have to follow suit, *i.e.*

$$\begin{aligned}
\psi^{(1)}_n &= i\left(\cos 2\theta\, \tilde{\psi}^{(1)}_{-n} + \sin 2\theta\, \tilde{\tilde{\psi}}^{(2)}_{-n}\right) & \qquad \bar{\psi}^{(1)}_n &= i\left(\cos 2\theta\, \tilde{\tilde{\psi}}^{(1)}_{-n} + \sin 2\theta\, \tilde{\psi}^{(2)}_{-n}\right) \\
\psi^{(2)}_n &= i\left(\cos 2\theta\, \tilde{\psi}^{(2)}_{-n} - \sin 2\theta\, \tilde{\tilde{\psi}}^{(1)}_{-n}\right) & \qquad \bar{\psi}^{(2)}_n &= i\left(\cos 2\theta\, \tilde{\tilde{\psi}}^{(2)}_{-n} - \sin 2\theta\, \tilde{\psi}^{(1)}_{-n}\right)
\end{aligned} \tag{4.34}$$

Given the explicit expressions for the $N = 2$ and $N = 4$ supercharges of the appendix (see (A.4) and (A.6)), we can now deduce the gluing conditions for the supercharges and the $\widehat{su}(2)_1$ currents. Explicitly we find that

$$\left(J^a_n + g\, \tilde{J}^a_{-n}\, g^{-1}\right) \|B1\rangle\rangle = 0\,, \qquad g = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} . \tag{4.35}$$

[Here we have chosen the convention that the Lie algebra generators are defined by

$$t^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \qquad t^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \qquad t^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .] \tag{4.36}$$

For all values of θ, the modified boundary state preserves the $N = 4$ superconformal algebra. However, we can ask whether there is an $N = 2$ subalgebra of the $N = 4$ algebra for which the gluing conditions are A-type. In particular, the $U(1)$ current K of this $N = 2$

subalgebra would then have to satisfy the gluing condition

$$\left(K_n - \tilde{K}_{-n}\right) \|B\rangle\rangle = 0\,. \tag{4.37}$$

Given (4.35) this means that K, regarded as an element of the Lie algebra of $su(2)$, satisfies

$$gKg^{-1} = -K\,. \tag{4.38}$$

It is easy to see that such a K only exists if $\cos 2\theta = 0$, *i.e.* for $2R_1R_2 = 1$. Thus unless $2R_1R_2 = 1$ (for $B_1 = B_2 = 0$), it is not possible to extend the boundary state $\|B1\rangle\rangle$ in this manner while preserving $N = 2$ A-type supersymmetry. On the other hand, if $2R_1R_2 = 1$ (for $B_1 = B_2 = 0$) is satisfied, one can easily see that the usual $N = 2$ subalgebra continues to satisfy an A-type gluing condition. A similar analysis also works for other modifications of the gluing conditions, as well as for $\|B2\rangle\rangle$.

These findings reflect now very nicely the results we obtained from the matrix factorisation point of view. There we saw that for generic deformations parametrised by a and b it was not possible to extend the (13)(24) and (14)(23) factorisations. However, there were special directions for which an extension was possible: for example, as was mentioned in section 3.5.1 we could extend the (13)(24) factorisation if a and b satisfy

$$a\eta_1^2\eta_2^2 + b = 0\,, \tag{4.39}$$

and a similar condition holds for the (14)(23) factorisations. In terms of the orbifold theory the condition (4.39) means that the two tori have the same Kaehler parameter (up to an $\mathrm{SL}(2,\mathbb{Z})$ transformation). This follows from the fact that the relation between a and the Kaehler parameter ρ_1 of the first torus is [29]

$$j(\rho_1) = \frac{1}{27\cdot 4}\frac{(a^2+12)^3}{(a^2-4)^2}\,, \tag{4.40}$$

with an identical relation between b and the Kaehler parameter ρ_2 of the second torus. In particular, the two Kaehler parameters only depend on a^2 and b^2, respectively, and thus (4.39) implies that $j(\rho_1) = j(\rho_2)$. This is in particular the case if $2R_1R_2 = 1$ (at $B_1 = B_2 = 0$). Thus the special unobstructed deformations correspond to each other.

It would be interesting to have a more precise dictionary between the different matrix factorisations (including the values of η_1 and η_2, *etc.*) and the orbifold D-branes. This would allow one to check these identifications in even more detail.

5. Conclusions

In this paper we have studied B-type D-branes on K3 from three different points of view: using geometrical methods (section 2), with the help of the matrix factorisation approach (section 3), and for the $T^4/\mathbb{Z}_4$ orbifold line in conformal field theory (section 4). We have shown that the results we obtained from these different points of view fit very well together. In particular, we have been able to understand the generic rank of the Picard lattice for K3's that are hypersurfaces in weighted projective space both from a geometrical point of

view and using matrix factorisation techniques. For the case of the $T^4/\mathbb{Z}_4$ orbifold line we have furthermore managed to identify in some detail the different matrix factorisation with boundary states in the orbifold conformal field theory. Furthermore, we could understand from both points of view why certain D-branes are obstructed against deformations of the bulk theory.

More generally, we have found a necessary criterion for when a given matrix factorisation can be analytically extended under a bulk deformation: this is only possible if the factorisation is uncharged under the RR field that corresponds to the bulk deformation. It would be good to understand this condition directly in conformal field theory.

Among other things, our results demonstrate convincingly that the matrix factorisation approach is a very powerful method to study D-branes at generic points in the moduli space where the traditional conformal field theory techniques are unavailable. One may hope to be able to push this further and deduce more global properties about D-branes on Calabi-Yau manifolds. This should, in particular, be possible for D-branes on K3 where we have extended supersymmetry.

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A. The RR ground states of the torus theory

In this appendix we collect various facts about the RR ground states of the torus orbifold. In particular, we exhibit the underlying $N = 2$ and $N = 4$ superconformal symmetry in the R sector of this theory. In the second subsection we explain the dictionary between the torus RR ground states and the corresponding states in the Gepner model in some detail.

A.1 The $N = 2$ and $N = 4$ algebras

To fix notation, let us first consider a single T^2 with $c = 3$. The left-moving (complex) bosonic and fermionic modes are denoted by α_m, $\bar{\alpha}_m$, ψ_n and $\bar{\psi}_n$. The bosonic modes satisfy the commutation relations

$$[\alpha_m, \alpha_n] = 0 = [\bar{\alpha}_m, \bar{\alpha}_n]\,, \qquad [\alpha_m, \bar{\alpha}_n] = m\delta_{m,-n}\,, \tag{A.1}$$

and the fermionic modes the anti-commutation relations

$$\{\psi_m, \psi_n\} = 0 = \{\bar{\psi}_m, \bar{\psi}_n\}\,, \qquad \{\psi_m, \bar{\psi}_n\} = \delta_{m,-n}\,. \tag{A.2}$$

The two (chiral) R ground states carry $N = 2$ quantum numbers $h = 1/8$ and $q = \pm 1/2$, and are mapped into one another by the action of the fermionic zero modes

$$\begin{aligned} \psi_0 \left|\frac{1}{8}, -\frac{1}{2}\right\rangle &= 0 \qquad & \psi_0 \left|\frac{1}{8}, \frac{1}{2}\right\rangle &= \left|\frac{1}{8}, -\frac{1}{2}\right\rangle \\ \bar{\psi}_0 \left|\frac{1}{8}, \frac{1}{2}\right\rangle &= 0 \qquad & \bar{\psi}_0 \left|\frac{1}{8}, -\frac{1}{2}\right\rangle &= \left|\frac{1}{8}, \frac{1}{2}\right\rangle . \end{aligned} \tag{A.3}$$

Here the $N = 2$ generators are defined, in terms of the free bosons and fermions, as (see for example [58])

$$\begin{aligned} L_n &= \sum_m : \alpha_{n-m}\bar{\alpha}_m : + \frac{1}{2}\sum_m (2m-n) : \bar{\psi}_{n-m}\psi_m : + \frac{1}{8}\,\delta_{n,0} \\ J_n &= \sum_m : \bar{\psi}_{n-m}\psi_m : - \frac{1}{2}\,\delta_{n,0} \\ G^+_n &= \sqrt{2}\sum_m \alpha_{n-m}\bar{\psi}_m \\ G^-_n &= \sqrt{2}\sum_m \bar{\alpha}_{n-m}\psi_m \,. \end{aligned} \tag{A.4}$$

One easily checks that they satisfy the correct $N = 2$ algebra with $c = 3$,

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \tfrac{c}{12}(m^3 - m)\delta_{m,-n} \\ [L_m, J_n] &= -nJ_{m+n} \\ \left[L_m, G^\pm_n\right] &= \left(\tfrac{1}{2}m - n\right) G^\pm_{m+n} \\ [J_m, J_n] &= \tfrac{c}{3} m \delta_{m,-n} \\ \left[J_m, G^\pm_n\right] &= \pm G^\pm_{m+n} \\ \left\{G^+_m, G^-_n\right\} &= 2L_{m+n} + (m-n)J_{m+n} + \tfrac{c}{3}(m^2 - \tfrac{1}{4})\delta_{m,-n} \\ \left\{G^+_m, G^+_n\right\} &= \left\{G^-_m, G^-_n\right\} = 0 \,. \end{aligned}$$

Furthermore, they have the correct $N = 2$ eigenvalues on the above states. [Note that the normal ordering for the fermions is defined by $: \bar{\psi}_m\psi_n : = \bar{\psi}_m\psi_n$ for $m \le n$ and $: \bar{\psi}_m\psi_n : = -\psi_n\bar{\psi}_m$ for $m > n$.] It is also obvious that the above states are annihilated by the zero modes $G^\pm_0$, as they must be.

For the case of interest to us, we have two such tori, and thus in fact an $N = 4$ algebra. We denote the relevant free field modes by $\alpha^{(i)}_n$ and $\psi^{(i)}_n$, where $i = 1, 2$. The additional generators of the $N = 4$ algebra are the generators $J^\pm_n$ that enhance the $u(1)$ current $J_n \equiv J^{(1)}_n + J^{(2)}_n$ to an $\widehat{su}(2)_1$ algebra

$$J^+_n = \sum_m : \bar{\psi}^{(1)}_{n-m}\,\bar{\psi}^{(2)}_m : , \qquad J^-_n = -\sum_m : \psi^{(1)}_{n-m}\,\psi^{(2)}_m : \,. \tag{A.5}$$

These generators are obviously orbifold invariant. Note that the normalisation of the $\widehat{su}(2)_1$ generators is slightly unusual: they satisfy

$$\begin{aligned} [J_m, J^\pm_n] &= \pm\, 2\, J^\pm_{m+n} \\ [J^+_m, J^-_n] &= J_{m+n} + m\,\delta_{m,-n} \\ [J_m, J_n] &= 2\,m\,\delta_{m,-n} \,. \end{aligned}$$

In addition we have two more supercharges $G'^{\pm}_n$ that are defined by

$$\begin{aligned} G'^{+}_n &= \sqrt{2}\sum_m \left(\bar\psi^{(1)}_{n-m}\,\bar\alpha^{(2)}_m - \bar\psi^{(2)}_{n-m}\,\bar\alpha^{(1)}_m\right) \\ G'^{-}_n &= \sqrt{2}\sum_m \left(\psi^{(1)}_{n-m}\,\alpha^{(2)}_m - \psi^{(2)}_{n-m}\,\alpha^{(1)}_m\right) . \end{aligned} \tag{A.6}$$

Together with $G^{\pm}_n \equiv G^{\pm(1)}_n + G^{\pm(2)}_n$ they then generate the $N = 4$ algebra [59]; in addition to the above commutation relation of the $N = 2$ generators we have

$$\begin{aligned} \{G^{\pm}_m, G'^{\pm}_n\} &= \mp 2(m-n)J^{\pm}_{m+n} & \{G^{\pm}_m, G'^{\mp}_n\} &= 0 \\ [L_m, G'^{\pm}_n] &= \left(\frac{m}{2} - n\right) G'^{\pm}_{m+n} & [J_m, G'^{\pm}_n] &= \pm G'^{\pm}_{m+n} \\ [J^{\pm}_m, G^{\pm}_n] &= [J^{\pm}_m, G'^{\pm}_n] = 0 \\ [J^{\pm}_m, G^{\mp}_n] &= \pm G'^{\pm}_{m+n} & [J^{\pm}_m, G'^{\mp}_n] &= \mp G^{\pm}_{m+n} \\ \{G'^{+}_m, G'^{-}_n\} &= 2L_{m+n} + (m-n)J_{m+n} + 2(m^2 - 1/4)\delta_{m,-n} . \end{aligned} \tag{A.7}$$

In the following we shall mostly work with the two $N = 2$ algebras corresponding to the two tori. The RR ground states are then characterised by their eigenvalues with respect to the two different U(1)-charges. We shall denote these states by $|\pm,\pm\rangle$, where

$$\begin{aligned} \psi^{(1)}_0|+,\pm\rangle &= |-,\pm\rangle & \psi^{(1)}_0|-,\pm\rangle &= 0 \\ \psi^{(2)}_0|\pm,+\rangle &= \mp|\pm,-\rangle & \psi^{(2)}_0|\pm,-\rangle &= 0 \\ \bar\psi^{(1)}_0|+,\pm\rangle &= 0 & \bar\psi^{(1)}_0|-,\pm\rangle &= |+,\pm\rangle \\ \bar\psi^{(2)}_0|\pm,+\rangle &= 0 & \bar\psi^{(2)}_0|\pm,-\rangle &= \mp|\pm,+\rangle . \end{aligned} \tag{A.8}$$

Note the signs in the second and fourth line — these are a consequence of the fact that the fermionic zero modes of the first and second torus anti-commute. In the full theory we then have such states for the left- and the right movers; these will be denoted by $|\pm,\pm\rangle \otimes \overline{|\pm,\pm\rangle}$. The action of the right-moving modes $\tilde\psi^{(j)}_0$ and $\tilde{\bar\psi}^{(j)}_0$ are given by the same relations as above. (Note though that the right-moving fermionic modes also anti-commute with the left-moving fermionic modes, and thus one has to be careful about relative signs!)

A.2 The identification with the Gepner model states

Having set up notation, we can now characterise the RR ground state $|0\rangle_{RR}$ that is annihilated by the modes

$$\psi^{-}_j|0\rangle_{RR} = \bar\psi^{-}_j|0\rangle_{RR} = 0\,, \qquad j = 1,2\,, \tag{A.9}$$

where

$$\psi^{\pm}_j = \frac{1}{\sqrt{2}}\left(\psi^{(j)}_0 \pm i\tilde\psi^{(j)}_0\right) , \qquad \bar\psi^{\pm}_j = \frac{1}{\sqrt{2}}\left(\bar\psi^{(j)}_0 \pm i\tilde{\bar\psi}^{(j)}_0\right) . \tag{A.10}$$

One easily convinces oneself that, in terms of the above basis,

$$|0\rangle_{RR} = \left(|+,+\rangle \otimes \overline{|-,-\rangle} - i|+,-\rangle \otimes \overline{|-,+\rangle} + i|-,+\rangle \otimes \overline{|+,-\rangle} - |-,-\rangle \otimes \overline{|+,+\rangle}\right) . \tag{A.11}$$

[In writing this formula we have chosen the convention that the state with $-$ is bosonic, while that with $+$ is fermionic; so we have for example

$$\tilde{\psi}_0^{(1)}|-,-\rangle\otimes\overline{|+,+\rangle}=|-,-\rangle\otimes\overline{|-,+\rangle}\,,\qquad \tilde{\psi}_0^{(1)}|-,+\rangle\otimes\overline{|+,+\rangle}=-|-,+\rangle\otimes\overline{|-,+\rangle}$$

etc.] Furthermore, we can write the various orbifold invariant states that are obtained by the action of the ψ^+ modes from this state in terms of the $N=2$ basis. One easily finds

$$\psi_1^+\psi_2^+|0\rangle_{RR}=-2|-,-\rangle\otimes\overline{|-,-\rangle}\,,\qquad \bar{\psi}_1^+\bar{\psi}_2^+|0\rangle_{RR}=-2|+,+\rangle\otimes\overline{|+,+\rangle}\,. \tag{A.12}$$

These two states therefore have $q=\bar{q}=-1$ and $q=\bar{q}=1$, respectively. Since the torus description is related by mirror symmetry to the Gepner model description, they must correspond to states in the Gepner model with $q=-\bar{q}=1$ and $q=-\bar{q}=-1$, respectively. These are precisely the RR ground states in the twisted $(n=1)$ and $(n=3)$ sectors, as we had already argued before.

For the other ground states we obtain

$$\begin{aligned}
|0\rangle_{RR} &= |+,+\rangle\otimes\overline{|-,-\rangle}-i|+,-\rangle\otimes\overline{|-,+\rangle}+i|-,+\rangle\otimes\overline{|+,-\rangle}-|-,-\rangle\otimes\overline{|+,+\rangle}\\
\bar{\psi}_1^+\psi_1^+|0\rangle_{RR} &= |+,+\rangle\otimes\overline{|-,-\rangle}-i|+,-\rangle\otimes\overline{|-,+\rangle}-i|-,+\rangle\otimes\overline{|+,-\rangle}+|-,-\rangle\otimes\overline{|+,+\rangle}\\
\bar{\psi}_2^+\psi_2^+|0\rangle_{RR} &= |+,+\rangle\otimes\overline{|-,-\rangle}+i|+,-\rangle\otimes\overline{|-,+\rangle}+i|-,+\rangle\otimes\overline{|+,-\rangle}+|-,-\rangle\otimes\overline{|+,+\rangle}\\
\bar{\psi}_2^+\psi_2^+\bar{\psi}_1^+\psi_1^+|0\rangle_{RR} &= |+,+\rangle\otimes\overline{|-,-\rangle}+i|+,-\rangle\otimes\overline{|-,+\rangle}-i|-,+\rangle\otimes\overline{|+,-\rangle}-|-,-\rangle\otimes\overline{|+,+\rangle}\,.
\end{aligned}$$

The A-type branes of the torus orbifold (that correspond to the B-type branes of the Gepner model) should therefore couple to the combinations

$$|\Psi_1^-\rangle=i\left(|-,+\rangle\otimes\overline{|+,-\rangle}-|+,-\rangle\otimes\overline{|-,+\rangle}\right)=\frac{1}{2}\left(|0\rangle_{RR}-\bar{\psi}_2^+\psi_2^+\bar{\psi}_1^+\psi_1^+|0\rangle_{RR}\right) \tag{A.13}$$

and

$$|\Psi_2^-\rangle=i\left(|-,+\rangle\otimes\overline{|+,-\rangle}+|+,-\rangle\otimes\overline{|-,+\rangle}\right)=\frac{1}{2}\left(\bar{\psi}_2^+\psi_2^+|0\rangle_{RR}-\bar{\psi}_1^+\psi_1^+|0\rangle_{RR}\right)\,. \tag{A.14}$$

This motivates our ansatz for the boundary states $|B1\rangle\!\rangle$ and $|B2\rangle\!\rangle$: the ground state of $|B1\rangle\!\rangle$ couples to $|\Psi_1^-\rangle$ (as well as to the states that correspond to $(n=1)$ and $(n=3)$), while the ground state of $|B2\rangle\!\rangle$ couples to $|\Psi_2^-\rangle$ (as well as again to the states that correspond to $(n=1)$ and $(n=3)$). The corresponding D-branes should therefore correspond to the (13)(24) and (14)(23) branes. As we have shown in the main part, D-branes with these gluing conditions are indeed obstructed under changing the Kaehler parameters of the two tori separately.

A.3 Spectral flow

The above analysis implies that the orbifold RR ground states that correspond to the two polynomials $x_1^2x_2^2$ and $x_3^2x_4^2$ are precisely the states

$$|-,+\rangle\otimes\overline{|+,-\rangle}\qquad\text{and}\qquad|+,-\rangle\otimes\overline{|-,+\rangle}\,. \tag{A.15}$$

Indeed, the other two states $|+,+\rangle\otimes\overline{|-,-\rangle}$ and $|-,-\rangle\otimes\overline{|+,+\rangle}$ couple only to B-type branes in the orbifold theory, and thus to A-type branes in the Gepner model.

On the other hand, the RR ground states corresponding to $x_1^2 x_2^2$ and $x_3^2 x_4^2$ must be the images under spectral flow of the NS-NS sector states that describe the deformation of the Kaehler parameters of the two tori. These are the states

$$\psi^{(1)}_{-1/2}\tilde{\bar{\psi}}^{(1)}_{-1/2}|0\rangle_{NSNS}\,, \qquad \bar{\psi}^{(1)}_{-1/2}\tilde{\psi}^{(1)}_{-1/2}|0\rangle_{NSNS} \tag{A.16}$$

and

$$\psi^{(2)}_{-1/2}\tilde{\bar{\psi}}^{(2)}_{-1/2}|0\rangle_{NSNS}\,, \qquad \bar{\psi}^{(2)}_{-1/2}\tilde{\psi}^{(2)}_{-1/2}|0\rangle_{NSNS}\,. \tag{A.17}$$

As a final consistency check of our identification we can now show that this is indeed the case. The spectral flow that defines a symmetry of the Gepner model acts symmetrically on left- and right-movers. In the torus orbifold, the corresponding flow should therefore act asymmetrically (since in the identification mirror symmetry has been performed). The first state in (A.15) has $h_1 = h_2 = \bar{h}_1 = \bar{h}_2 = 1/8$ and $q_1 = -1/2$, $q_2 = 1/2$, $\bar{q}_1 = 1/2$, $\bar{q}_2 = -1/2$. Under spectral flow by one half unit it therefore flows to a NS-NS state with the quantum numbers $h_1 = 1/2$, $\bar{h}_1 = 1/2$, $h_2 = 0$, $\bar{h}_2 = 0$, $q_1 = -1$, $\tilde{q}_1 = 1$, $q_2 = 0$, $\tilde{q}_2 = 0$. This is then precisely one of the states in (A.16). The analysis for the other states is similar.

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